Method of Solving Extension of Interval in Assignment Problem

Dr. S. A. Muthia, S. Lakshmi, S. Narmatha
Assistant Professor

Ramanujam Centre for Higher Mathematics, Alagappa University, Karikudi, India
Department of Mathematics (PG & UG-SF), PSGR Krishnamma College for Women, Coimbatore, India

Abstract:
In this paper we introduce a Method of Solving Extension of Interval [MSEI]. Some special cases in assignment problem and its application have been discussed. Compared the optimal solution of Hungarian method for both maximization and minimization type.

Keywords: Assignment Problem, Hungarian assignment method, MOIAM, Optimization

Introduction: The assignment problem is a special type of linear programming problem where assignees are being assigned to perform tasks. The assignees might be employers who need to be given work assignments. Assigning people to jobs is a common application of assignment problem. However the assignees need not be people. They also could be machine of vehicles or plants or even time slots to be assigned tasks. So for in the literature, there are mainly four methods namely Enumeration method, Simplex method, Transportation problem and Hungarian method. Hungarian method is one of the best methods available for solving an assignment problem. There are so many alternative methods are available. One of these is “Solving One’s Interval Linear Assignment Problem”. We introduce the new “Method of Solving Extension of Interval in Assignment Problem”. Where $C_{ij}$ is the cost or measure of effectiveness of assigning $i$th job to the $j$th facility. An assignment plan is optimal if it also minimizes (optimizes) the total cost of doing all the jobs.

Mathematical formulation of assignment problem: In the each assignment problem there is a matrix called the cost matrix $[C_{ij}]$ where $C_{ij}$ is the cost or measure of effectiveness of assigning $i$th job to the $j$th facility. An assignment plan is optimal if it also minimizes (optimizes) the total cost of doing all the jobs.

Facilities (Destinations)

| $f_1$ | $f_2$ | $f_3$ | ... | $f_k$ | $a_i$
|------|------|------|-----|------|------
| $J_1$ | $c_{11}$ | $c_{12}$ | ... | $c_{1k}$ | $1$
| $J_2$ | $c_{21}$ | $c_{22}$ | ... | $c_{2k}$ | $1$
| $J_k$ | $c_{k1}$ | $c_{k2}$ | ... | $c_{kk}$ | $1$

Jobs

To minimize the total cost, distribution made for $k$ units to the $k$ destination requiring only 1 unit.

Let $x_{ij} = \begin{cases} 1, & \text{if } i\text{th job is assigned to } j\text{th facility} \\ 0, & \text{if } i\text{th job is not assigned to } j\text{th facility} \end{cases}$

Determine $x_{ij}, \quad i,j=1,2,\ldots,m$ so as to minimize

$Z=\sum_{i=1}^{k}\sum_{j=1}^{k}c_{ij}x_{ij}$

$\sum_{i=1}^{k}x_{ij} = 1, \quad j = 1, 2, \ldots, k$

$\sum_{j=1}^{k}x_{ij} = 1, \quad i = 1, 2, \ldots, k$ and $x_{ij} = 0 \text{ or } 1$

In the interval form the left value is denoted by $[x]$ and the right value is denoted by $[y]$. Let $[x, x]$ and $[y, y]$ be two elements then the following arithmetic are well known

(i) $[x, x] + [y, y] = [x + y, x + y]$

(ii) $[x, x] \times [y, y] = [\min \{x, y\}, \max \{x, y\}]$

(iii) $[x, x] ÷ [y, y] = [\min \{x, y\}, \max \{x, y\}] = [\min \{x+y, x+y\}, \max \{x+y, x+y\}]$ provide if $[y, y] \neq [0,0]$

Procedure for Hungarian method

Step 1: Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (column) with zero cost elements.

Step 2: Subtract the minimum element in each row from all the elements of the respective rows.

Step 3: Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

Step 4: Then draw minimum number of horizontal and vertical lines to cover all zeros in the resulting matrix. Let the minimum number of lines be $N$. Now there are two possible cases.

Case (a): If $N=n$, where $n$ is the order of matrix, then an optimal assignment can be made. So make the assignment to get the required solution.

Case (b): If $N<n$, then proceed to step 5.

Step 5: Determine the smallest uncovered element in the matrix (element not covered by $N$ lines). Subtract this minimum element from all uncovered element and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step 6: Repeat step (3) and step (4) until we get the case (a) of step 4.

Step 7: (To make zero assignment) Examine the rows successively until a row-wise exactly single zero is found. Circle (0) this zero to make the assignment. Then mark a cross ($\times$) over all zeros if lying in the column of the circled zero, showing that they can’t be considered for future assignment. Continue in this manner until all the zeros have been examined. Repeat the same procedure for column also.

Step 8: Repeat the step 6 successively until one of the following situation arises (i) If no unmarked zero is left, then the process ends (or) (ii) If there lies more than one of the unmarked zero in any column or row then, circle one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row or column. Repeat the process until no unmarked zero is left in the matrix.

Step 9: Thus exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeros will give the optimal assignment.
**Balanced Assignment Problem:** when the number of rows and column are equal in the cost matrix then the given problem is a balanced assignment problem.

**Example 1:** Consider the assignment problem.

<table>
<thead>
<tr>
<th>Machine</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jobs</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>7</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

Assign the four jobs to the machine to minimize the total cost.

**Solution:** Cost matrix Row reduction

Assign the four jobs to the machine to minimize the total cost.

**Unbalanced Assignment Problem**

The number of row and column are not equal then it is called an unbalanced assignment problem. In this case convert it in to balanced assignment problem by adding dummy row (s) or column(s) with the cost values of zeros (0). Then solve the problem by Hungarian method.

**Example 2:** The coach of an age group swim team needs to assign swimmers to a 200-yard medley relay team to send to the junior Olympics. Since most of his best swimmers are very fast in more than one stroke, it is not clear which swimmer should be assigned to each of the four strokes. The five swimmers and best times (in seconds) they have achieved in each of the strokes (for 50 yards) are

<table>
<thead>
<tr>
<th>Stroke</th>
<th>David</th>
<th>Shiva</th>
<th>Tony</th>
<th>Rahul</th>
<th>Abdulla</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backstroke</td>
<td>37.7</td>
<td>32.9</td>
<td>33.8</td>
<td>37.0</td>
<td>35.4</td>
</tr>
<tr>
<td>Breaststroke</td>
<td>43.4</td>
<td>33.1</td>
<td>42.2</td>
<td>34.7</td>
<td>41.8</td>
</tr>
<tr>
<td>Butterfly</td>
<td>33.3</td>
<td>28.5</td>
<td>38.9</td>
<td>30.4</td>
<td>33.6</td>
</tr>
<tr>
<td>Freestyle</td>
<td>29.2</td>
<td>26.4</td>
<td>29.6</td>
<td>28.5</td>
<td>31.1</td>
</tr>
</tbody>
</table>

The coach wishes to determine how to assign four swimmers to the four different strokes to minimize the sum of the corresponding best times. Formulate the problem as an assignment problem and also obtain an optimal solution.

**Solution:** Given problem is unbalanced type it is converted to the balanced problem by adding one row with the cost values are 0. The existing cost matrix is

<table>
<thead>
<tr>
<th>BACK</th>
<th>BREAST</th>
<th>BUTTER</th>
<th>FREE</th>
<th>DUMMY</th>
</tr>
</thead>
<tbody>
<tr>
<td>37.7</td>
<td>43.4</td>
<td>33.3</td>
<td>39.2</td>
<td>0</td>
</tr>
<tr>
<td>32.9</td>
<td>33.1</td>
<td>28.5</td>
<td>26.4</td>
<td>0</td>
</tr>
<tr>
<td>33.8</td>
<td>42.2</td>
<td>38.9</td>
<td>29.6</td>
<td>0</td>
</tr>
<tr>
<td>37.0</td>
<td>34.7</td>
<td>30.4</td>
<td>28.5</td>
<td>0</td>
</tr>
<tr>
<td>35.4</td>
<td>41.8</td>
<td>33.6</td>
<td>31.1</td>
<td>0</td>
</tr>
</tbody>
</table>

The optimal solution is David → Freestyle, Shiva → Butterfly, Tony → Backstroke, Rahul → Breaststroke and Abdulla is rejected.
Restriction Method: If we don’t know the cost of any particular fixation of the machine cost can be taken in this type.

Example 3: Assign the machine to the appropriate place which gives the minimum total elapsed time.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>M2</td>
<td>12</td>
<td>9</td>
<td>∞</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>M3</td>
<td>-</td>
<td>11</td>
<td>14</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>M4</td>
<td>14</td>
<td>8</td>
<td>12</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

Solution: The given problem is unbalanced and restriction type. Add a dummy row and consider the not given value as the highest expense to fix that particular machine to that place.

Using Hungarian method the cost matrix is given below.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>7</th>
<th>14</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>17</td>
<td>22</td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>17</td>
<td>22</td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>22</td>
<td>26</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

Reduction in row wise Reduction in column wise

Salesman A assign to area I, Salesman B assign to II, Salesman C assign to III, Salesman D assign to IV to get the optimal solution. The maximum total expected sales is 42+25+20+12=99

Procedure for Method of Solving Extension of Interval In Assignment Problem [MSEI]

Step 1: Fix the interval for all the cost values by subtract 2 from the given cost as the left value. The right values of the interval find by add 2 from the given cost. (Note: The left value should not be non-negative. For this case write as a zero when we get the negative)

Step 2: After fixing the interval for the cost matrix. Find the smallest interval cost for the each row in the interval assignment matrix and subtract each element of that row. This results at least one zero in each rows. Repeat this Step for the column wise also.

Step 3: Draw the minimum number of lines to cover all the zero intervals. If the number of drawn lines < n, then the complete assignment is not possible go to step 4. If the number of lines is equal to n then we get the optimum assignment.

Step 4: Find the smallest interval which are not covered by any lines, subtract this interval to all non-covered cost values and add it to the crossing lines intervals.

Step 5: Repeat the step 2 to step 4 until an optimal solution is obtained.

Balanced assignment problem: Example 1 can be solved by MSEI method

Solution: Cost matrix Cost interval matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>4</th>
<th>6</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>7</td>
<td>10</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>11</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>8</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>[0.3]</th>
<th>[2.6]</th>
<th>[4.8]</th>
<th>[1.5]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7,11]</td>
<td>[5.9]</td>
<td>[8,12]</td>
<td>[7.11]</td>
<td></td>
</tr>
<tr>
<td>[2,6]</td>
<td>[3.7]</td>
<td>[9,13]</td>
<td>[5.9]</td>
<td></td>
</tr>
<tr>
<td>[6,10]</td>
<td>[5.9]</td>
<td>[6,10]</td>
<td>[3.7]</td>
<td></td>
</tr>
</tbody>
</table>
Reduction of interval in row wise

\[
\begin{bmatrix}
0 & 2.3 & 4.5 & 1.2 \\
2.2 & 0 & 3.3 & 2.2 \\
0 & 1.1 & 7.7 & 3.3 \\
3.3 & 2.2 & 3.3 & 0
\end{bmatrix}
\]

Reduction of interval in column wise

\[
\begin{bmatrix}
0 & 2.3 & 1.2 & 0.1 \\
3.3 & 0.0 & 0.0 & 2.2 \\
0 & 0.0 & 3.3 & 2.2 \\
4.4 & 2.2 & 0.0 & 0.0
\end{bmatrix}
\]

To get the optimal solution we must assign 1→A, 2→C, 3→B, 4→D. the minimum cost to fit the machine is 1+10+5+5=21.

**Unbalanced Assignment Problem:** The number of row and column are not equal then it is called an unbalanced assignment problem. In this case convert it in to balanced assignment problem by adding dummy row(s) or column(s) with the cost values of zeros (0). Then solve the problem by MSEI method.

Using MSEI method we can find the optimal solution for **Example 2**

**Solution:** The existing cost matrix is

\[
\begin{array}{cccc}
37.7 & 43.4 & 33.3 & 39.2 & 0 \\
32.9 & 33.1 & 28.5 & 26.4 & 0 \\
33.8 & 42.2 & 38.9 & 29.6 & 0 \\
37.0 & 34.7 & 30.4 & 28.5 & 0 \\
35.4 & 41.8 & 33.6 & 31.1 & 0
\end{array}
\]

Cost interval matrix

\[
\begin{bmatrix}
35.7,39.7 & 41.4,45.7 & 31.3,35.3 & 27.3,31.3 & 0,0 \\
7,4 & 3,3 & 2,2 & 1,1 & 1,1 \\
30.9,34.1 & 31.1,35.1 & 26.5,30.1 & 24.4,28.1 & 0,0 \\
9,1 & 1,1 & 5,4 & 1,1 & 1,1 \\
31.8,35.4 & 40.2,44.2 & 36.9,40.2 & 27.6,31.2 & 0,0 \\
8,2 & 9,9 & 6,6 & 1,1 & 1,1 \\
35.9,35.9 & 32.7,36.7 & 28.4,32.6 & 26.5,30.6 & 0,0 \\
7,7 & 4,4 & 5,5 & 1,1 & 1,1 \\
33.4,37.4 & 39.8,43.1 & 31.6,35.1 & 29.1,33.1 & 0,0 \\
4,4 & 8,8 & 6,6 & 1,1 & 1,1
\end{bmatrix}
\]

The optimal solution is David → Freestyle, Shiva → Butterfly, Tony→Backstroke, Rahul→Breaststroke and Abdulla is rejected.
Restriction method:

Example 3 can be solved by MSEI method.

Solution: The cost matrix is

\[
\begin{bmatrix}
9 & 11 & 15 & 10 & 11 \\
12 & 9 & \infty & 10 & 9 \\
\infty & 11 & 14 & 11 & 7 \\
14 & 8 & 12 & 7 & 8 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Cost interval matrix

\[
\begin{bmatrix}
[0,0] & [0,0] & [0,0] & [0,0] & [0,0]
\end{bmatrix}
\]

The reduction in row and column wise the optimal matrix is

\[
\begin{bmatrix}
[3,3] & [0,0] & \infty & [1,1] & \infty \\
\end{bmatrix}
\]

Machine 1 is fixed in place A. Machine 2 is fixed in place B. Machine 3 is fixed in place E. Machine 4 is placed in D. Machine 5 is placed in C. Total minimum cost to fix these machines are 9+9+11+12 = 41hrs

Maximization Method:

Example 4 can be solved by MSEI method.

Solution: The cost matrix is

\[
\begin{bmatrix}
0 & 7 & 14 & 21 \\
12 & 17 & 22 & 27 \\
12 & 17 & 22 & 27 \\
18 & 22 & 26 & 30
\end{bmatrix}
\]

Cost interval matrix

\[
\begin{bmatrix}
\end{bmatrix}
\]

Reduction in row wise

\[
\begin{bmatrix}
[0,0] & [4,4] & [8,8] & [12,12] \\
\end{bmatrix}
\]

Reduction in column wise

\[
\begin{bmatrix}
[0,0] & [1,3] & [4,6] & [7,9] \\
[0,0] & [1,1] & [2,2] & [3,3] \\
[0,0] & [1,1] & [2,2] & [3,3] \\
[0,0] & [0,0] & [0,0] & [0,0]
\end{bmatrix}
\]

The optimal matrix is given below. Sales man A assign to area I, Sales man B assign to II, Sales man C assign to III, Salesman D assign to IV to get the optimal solution. The maximum total expected sales is 42+25+20+12=99

Conclusion: We concluded that optimal solution of both Hungarian method and MSEI method are same in various cases.

<table>
<thead>
<tr>
<th>Problem type</th>
<th>Balance (example 1)</th>
<th>Unbalanced (example 2)</th>
<th>Restriction (example 3)</th>
<th>Maximization (example 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hungarian method</td>
<td>21</td>
<td>127.2</td>
<td>41</td>
<td>99</td>
</tr>
<tr>
<td>MSEI method</td>
<td>21</td>
<td>127.2</td>
<td>41</td>
<td>99</td>
</tr>
</tbody>
</table>

We can apply MSEI method for all types of assignment problem is verified and compared with the Hungarian method optimal solution.

References:


