On Centre Fixed Vertex Covering Number and Polynomial of Some Standard Graphs

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Abstract:
The Vertex Cover Polynomial of a graph G of order n has been already introduced in [3]. It is defined as the polynomial C(G, x) = ∑ c(G, i)x^i, where c(G, i) is the number of vertex covering sets of G of size i and β(G) is the covering number of G. In this paper, we introduced the concept of Fixed Cover Polynomial of a graph G, and studied the properties of some standard graphs of order ‘n’.

Key Words: Vertex Covering, Vertex covering number, Vertex covering Polynomial, Centre.

I. INTRODUCTION

Let G = (V, E) be a finite, undirected, nontrivial and connected simple graph. For any vertex v ∈ V, the open neighborhood of v is the set N(v) = {u ∈ V / uv ∈ E} and the closed neighborhood of v is the set N[v] = N(v) ∪ {v}. For a set S ⊆ V, the open neighborhood of S is N(S) = ∪v∈S N(v) and the closed neighborhood of S is N[S] = N(S) ∪ S. A set K ⊆ V is a vertex covering of G, if every edge uv ∈ E is adjacent to at least one vertex in K. The vertex covering number β(G) is the minimum cardinality of the vertex covering sets in G. A vertex covering set of cardinality β(G) is called a β-set. The eccentricity of a vertex u ∈ V is denoted by e(u) and is defined by e(u) = max {d(u, v) | for any v ∈ V}. Let the centre of G be denoted by C(G) and is defined as the set of all vertices u ∈ V such that e(u) is minimum. Let M = C(G); a set F ⊆ V is a fixed covering of G, if every edge uv ∈ E is adjacent to at least one vertex in F and F ∩ M ≠ ∅. The fixed covering number f(G) is the minimum cardinality of the fixed covering sets in G. A fixed covering set with cardinality f(G) is called a f-set. Let F(G, i) be the family of all fixed covering sets in G, with cardinality i. Let f(G, i) = |F(G, i)|. The polynomial F(G, x) = ∑ f(G, i)x^i is defined as the fixed cover polynomial of G.

In [7] many properties of the vertex cover polynomials of some standard graphs such as Kn, Pn, Cn, Kmn etc have been studied.

II. FIXED COVER POLYNOMIAL

2.1. Definition

A graph G is said to be complete if any pair of vertices u, v ∈ V are adjacent in G. A complete graph with n vertices is denoted by Kn.

2.2. Definition

A path is a connected graph with two pendant vertices and all other vertices are of degree two. A path with n vertices is denoted by Pn.

2.3. Definition

A closed path is called a cycle; that is a path which is originating and ending with the same vertex. A cycle with n vertices is denoted by Cn.

2.4. Definition

A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle. The new edges are called the spokes of the wheel.

2.5. Definition

A bipartite graph is a graph, whose vertices can be partitioned into two non empty sets A and B, with |A| = m and |B| = n such that every edge of G has one end with an element of A and other end with the element of B. A complete bipartite graph with m + n vertices is denoted by Knm.

K1,n is said to be star graph.

2.6. Definition

A Lollipop graph is obtained by connecting a new vertex with any one of the vertex of Kn.

2.7. Definition

The composition of K1 and Kn is called one corona, that is connecting one pendant vertex to each vertex of a complete graph with n vertices is called one corona.
2.8. Definition

The composition of $K_2$ and $K_n$ is called 2-corona.

2.9. Definition

Barbell graph is a simple graph obtained by connecting two copies of $K_n$ by a bridge and is denoted by $B_n$.

2.1. Lemma

The fixed cover polynomial of $K_n$ is

$$F(G, x) = x^{n-1} (n + x).$$

Proof

Here $C(G) = \{v_0, v_1, v_2, \ldots, v_n\}$

Therefore, every vertex covering set is a fixed covering set.

The minimum cardinality of fixed covering sets are $n - 1$.

There are $n$ such sets.

Therefore, $F(K_n, x) = \sum_{i=0}^{n-1} F(G, i) x^i = n x^{n-1} + x^n = x^{n-1} (n + x)$.

2.2. Lemma

The fixed cover polynomial of the star graph $G = K_{1,n}$ is

$$F(G, x) = x \prod_{i=1}^{n}(1 + x)^{i}.$$

Proof

Let the vertices of $K_{1,n}$ be denoted by

$S = \{v_0, v_1, v_2, \ldots, v_n\}$, where $d(v_0) = n$ and $d(v_i) = 1, i = 1 \ldots n$

We have $M = \{v_0\}$ and $f(G) = 1.$

$F(G, 2) = \{\{v_0, v_1\}, \{v_0, v_2\} \ldots \{v_0, v_n\}\}$

Therefore, $F(G, 2) = n C_1$

$F(G, 3) = \{\{v_0, v_i, v_j\} | i, j = 1 \ldots n : i \neq j\}$

Therefore, $F(G, 3) = 3 n C_2$.

Proceeding this way we get $F(G, n - 1) = n C_{n-2}$

$F(G, n) = n C_{n-1}$ and $f(G, n + 1) = 1$.

Therefore, the fixed cover polynomial of $K_{1,n}$ is

$$F(G, x) = x + n c_1 x^2 + n c_2 x^3 + \ldots + n c_{n-1} x^{n} + x^{n+1}$$

$$= x \prod_{i=1}^{n}(1 + x)^{i}.$$ 

2.3. Theorem

The fixed cover polynomial of lolly pop graph $G = L_{1,n}$ is $F(G, x) = x^{n-1} [(n-1) + n x + x^2].$

Proof

Let $S = \{v_0, v_1, v_2, \ldots, v_n\}$ be the vertex set of $G.$

Figure 1

We have $d(v_0) = 1, d(v_i) = n$ and $d(v_j) = n - 1, i = 2 \ldots n$

Clearly $C(G) = \{v_1\}$.

The fixed vertex covering sets with minimum cardinality are $\{v_0, v_1, v_3, v_4, \ldots, v_n\}, \{v_1, v_2, v_4, \ldots, v_n\}, \{v_1, v_2, v_3, v_5, \ldots, v_n\}, \{v_1, v_2, v_3, \ldots, v_{n-1}\}.$

Therefore, $f(G, n - 1) = n - 1$.

The fixed covering sets with cardinality $n$ are

$F(G, n) = \{S \{v_0\}, S \{v_2\}, \ldots S \{v_n\}\}$.

Therefore, $f(G, n) = n$ and $f(G, n + 1) = 1$.

Therefore, the fixed cover polynomial of $L_{1,n}$ is

$$F(G, x) = (n - 1)x^{n-1} + n x^n + x^{n+1}$$

$$= x^{n-1} (x^2 + n x + n - 1).$$

2.4. Lemma

The fixed cover polynomial of the wheel $W_{1,n}$ is

$$F(W_{1,n}, x) = \sum_{i=\lceil n/2 \rceil}^{n} n-i \left(\begin{array}{c} i+1 \\ n-i \end{array}\right) x^i.$$

Proof

The number of vertex cover sets of cardinality $i$ is the same as the number of centre fixed sets of cardinality $i = 1, 2, \ldots, n-1$

When $i = n,$ $C(W_{1,n}, n) = \{S \{v_i\}, i = 1, 2, \ldots, n\}$

Here $C(W_{1,n}, n) = f(W_{1,n}, n).$

Hence $F(W_{1,n}, x) = \sum_{i=\lceil n/2 \rceil}^{n} n-i \left(\begin{array}{c} i+1 \\ n-i \end{array}\right) x^i.$

2.5. Theorem

The fixed cover polynomial of the barbell graph $G$ is

$$F(G, x) = x^{2n} [n^2 + 2n + 1].$$

Proof

Let the vertex set of $G$ be denoted by

$V = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$

Let $S_1 = \{v_1, v_2, \ldots, v_n\} ; S_2 = \{u_1, u_2, \ldots, u_n\}$
The fixed covering sets with cardinality 2n + 1 are
\[ F(G, 2n + 1) = \{V - \{v_i\} / i = 1, 2, \ldots n, i \neq j \}. \]
Therefore, \( f(G, 2n + 1) = 1 \).

Proceeding this way,
\[
\begin{align*}
F(G, n + 2) &= nC_n + n((n - 1)C_{n - 1}) \\
F(G, n + 4) &= nC_n + n((n - 1)C_{n - 1}) + \ldots + nC_n + n((n - 1)C_{n - 1}) \\
&= nC_n + n((n - 1)C_{n - 1}) + \ldots + nC_n + n((n - 1)C_{n - 1}) \\
&= nC_n + n((n - 1)C_{n - 1}) + \ldots + nC_n + n((n - 1)C_{n - 1})
\end{align*}
\]
That is,
\[
F(G, x) = \sum_{r = 0}^{n-1} \left\{ nC_r + n((n - 1)C_{r - 1}) \right\} x^{n+1} + \ldots + nC_n + n((n - 1)C_{n-1}) x^{n+1} + \ldots + nC_n + n((n - 1)C_{n-1}) x^{n+1}.
\]

2.7. Theorem

The fixed cover polynomial of \( K_n \circ K_2 \) is
\[
F(G, x) = \sum_{i = 0}^{n} \left\{ nC_i + 2(n - 1)(n - 1)C_i \right\} x^{n+i} + \ldots + nC_n + n((n - 1)C_{n-1}) x^{n+1}.
\]

Proof

Let \( V = \{v_1, v_2, \ldots v_n, u_1, u_2, \ldots u_n\} \) be the vertex set of \( G \). The vertices of \( V \) can be partitioned into three sets \( S_1, S_2, \) and \( S_3 \) such that
\[
S_1 = \{v_1, v_2, \ldots v_n\} \quad \text{and} \quad S_2 = \{u_1, u_2, \ldots u_n\} \quad \text{and} \quad S_3 = \{w_1, w_2, \ldots w_n\}.
\]
Note that the induced subgraph with vertex set \( S_1 \) is complete and the induced subgraphs with vertices \( \{v_i, u_i, w_i\}, i = 1, 2, \ldots n \) are also complete.

The fixed covering sets with minimum cardinality 2n are
\[
S_1 \cup S_2 \cup S_3 \cup \{v_i, u_i, w_i\}, i = 1, 2, \ldots n.
\]
Therefore, \( f(G, n + 2) = nC_n + n((n - 1)C_{n - 1}) \).

Figure 3

The fixed covering sets with minimum cardinality are
\[
F(G, n) = \{S_1 \cup \{v_i\} \cup \{u_i\} / i = 1, 2, \ldots n\}.
\]
Therefore, \( f(G, n) = n + 1 \).

The fixed covering sets with cardinality n + 1 are
\[
F(G, n + 1) = \{S_1 \cup \{v_i\} / i = 1, 2, \ldots n\} \cup \{S_1 \setminus \{v_i\} \cup \{u_i, u_j\} / i, j = 1, 2, \ldots n, i \neq j\}.
\]
Therefore, \( f(G, n + 1) = nC_n + n((n - 1)C_{n - 1}) \).

The fixed covering sets with cardinality (n + 2) are
\[
\{S_1 \cup \{v_i, u_j\} / i, j = 1, 2, \ldots n, i \neq j\}.
\]
Therefore, 
\[ f(G, 2n) = 2n + [nc_0 + nc_1 + \ldots + nc_n] \]
The covering sets with cardinality \(2n + 1\) are of the form
\[ S_1 \cup S_2 \cup \{w_i \} \land i = 1 \ldots n ; \ S_1 \cup S_2 \cup \{v_i \} \]
\[ \cup \{w_i, w_j\}, i \neq j \} \]
\[ S_1 \cup S_2 \cup \{v_i \} \cup \{w_i, w_j\}, i, j = 1 \ldots n, i \neq j; \]
\[ S_1 \cup S_2 \cup \{v_i \} \cup \{w_i, w_j\}, i, j = 1 \ldots n, i \neq j; \]
\[ S_1 \cup S_2 \cup \{u_i, u_j\} \cup \{w_i, w_j, w_k\}, i, j, k = 1 \ldots n, i \neq j \neq k; \]
\[ S_1 \cup S_2 \cup \{u_i, u_j, u_k\} \cup \{w_i, w_j, w_k, w_m\}, i, j, k, m = 1 \ldots n, i \neq j \neq k \neq m \}

The covering sets with cardinality \(2n + 2\) are of the form,
\[ S_1 \cup S_2 \cup \{w_i \} \land i = 1 \ldots n, i \neq j \}
\[ S_1 \cup S_2 \cup \{v_i \} \cup \{w_i, w_j\}, i, j = 1 \ldots n, i \neq j \neq k; \]
\[ S_1 \cup S_2 \cup \{v_i \} \cup \{w_i, w_j\}, i, j, k = 1 \ldots n, i \neq j \neq k; \]
\[ S_1 \cup S_2 \cup \{u_i, u_j\} \cup \{w_i, w_j, w_k\}, i, j, k = 1 \ldots n, i \neq j \neq k; \]
\[ S_1 \cup S_2 \cup \{u_i, u_j, u_k\} \cup \{w_i, w_j, w_k, w_m\}, i, j, k, m = 1 \ldots n, i \neq j \neq k \neq m \}

Therefore,
\[ f(G, 2n + 1) = n + n [(n - 1) C_1] + n [(n - 1) C_1]
\ + n C_1 [(n - 1) C_1] \]
\[ + n C_2 [(n - 2) C_1] \]
\[ + \ldots \]
\[ + n C_{n-1} [(n - n - 1) C_1] \]
\[ \ldots \]


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