Diffusion Process for an G/G/m/N Queue with Priority

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Abstract:
This paper deals with finite capacity queueing system with priority classes. We develop stable G/G/m/N model with general inter-arrival and service time distributions and parallel servers (m ≥ 1) under pre-emptive resume (PR) scheduling. By applying refined diffusion approximation technique with elementary reflecting boundary, approximate formula for the number of customers in the system, delay probability and mean queue length are established.

Kew Words: Diffusion process, Elementary returns boundary, Priority, Approximation technique. Mean queue length.

INTRODUCTION
With the increase in complexity of man-made system, performance modeling of flexible manufacturing system, production system, computer system, communication system, etc. has received attention of many researches. The priority discipline is also common to model these system based on queue theoretic approach. The pre-emptive resume priority scheduling is very important for the study of computer systems and packet data networks with parallel processors where jobs and packets are subject to transmitted. We consider a stable G/G/m/N (PR) queue with general inter-arrival and service time distributions. This paper analyses the effect of pre-emptive resume rule under R priority classes.


The purpose of present paper is to investigate finite capacity G/G/m/N queue with priority classes using diffusion approximation technique. We impose two reflecting barriers at 0 and N to provide solution to diffusion equations retaining the piecewise continuity of the infinitesimal mean and variance. We have also modified the boundary condition and apply the solution approach suggested Sharma and Jain [1998].

2. THE MODEL
The G/G/m queue with R(≥ 2) priority classes under pre-emptive resume (PR) discipline is considered. It is assumed that class r jobs, r = 1, 2, 3, ..., R arrive according to Poisson fashion with mean rate λ_r and square coefficient of variance C^2_{ar}. Customers are served by one of the m (≥ 2) parallel servers according to of pre-emptive resume priority rule. The service times of the customers are independent and identically distributed random variables with mean rate 1/μ_r and square coefficient of variance C^2_{sr}. Suppose that the priority classes are indexed from 1 to R in decreasing order of priority i.e. for r < s, the jobs in priority r are given preference over the jobs in class s; and the jobs of the same class are served according to FCFS rule. It is assumed that the system reaches a steady state so that the traffic intensity satisfies
\[ \sum_{r=1}^{R} \frac{\lambda_r}{\mu_r} < m \]  

\[ (1) \]

Let \( \Lambda_r \) and \( \bar{\mu}_r \) be the overall arrival rate and service rate; \( \bar{C}_{ar} \) and \( \bar{C}_{sr} \) be the overall square coefficient of variance of inter-arrival time and service time, respectively. Denote by \( R_r \) is the mean response time of class \( r \) jobs. Let \( \bar{R}(r) \) denote the overall mean response time. For the first \( r \) priority classes of jobs, \( r \in [1, R] \), \( \Lambda_r \) and \( \bar{\mu}_r \), are given by

\[ \Lambda_r = \sum_{s=1}^{r} \lambda_s, r = 1,2,3,\ldots,R \]  

\[ (2) \]

and

\[ \bar{\mu}_r = \frac{\Lambda_r}{\sum_{s=1}^{r} \mu_s} \]  

\[ (3) \]

\( \bar{C}_{ar} \) is determined by the merging formula of \( r \) - GE streams, and is obtained as

\[ \bar{C}_{ar} = 1 + \left\{ \sum_{s=1}^{r} \frac{\lambda_s}{\Lambda_s} \left( \bar{C}_{as} + 1 \right)^{-1} \right\}^{-1} \]  

\[ (4) \]

By applying the law of total expectations, \( \bar{C}_{sr} \) is given by

\[ \bar{C}_{sr} = \bar{\mu}_r^{-2} \left( \sum_{l=1}^{r} \frac{\lambda_l}{\bar{\mu}_l} \left( \bar{C}_{sl} + 1 \right)^{-1} \right) \]  

\[ (5) \]

We propose

(i) for \( k-1 < x \leq k \) \( (k = 1,2,3,\ldots,m) \)

\[ b(x) = \lambda_x - \left( \left[ X \right] \wedge m \right) \bar{\mu}_r \equiv b_k \]  

\[ (7) \]

\[ a(x) = \lambda_x \left( \bar{C}_{ar} \bar{\mu} + \sigma_x^2 \right) + \left( \left[ X \right] \wedge m \right) \bar{\mu}_r \bar{C}_{sr} \equiv a_k \]  

\[ (8) \]

and(ii) for \( m < x < N \)

\[ b(x) = \lambda_x - \left( m \bar{\mu} - \left[ X - m \right] \right) \equiv b_k \]  

\[ (9) \]

\[ a(x) = \lambda_x \left( \bar{C}_{ar} \bar{\mu} + \sigma_x^2 \right) + \left( m \bar{\mu}_r + \left[ X - m \right] \nu \right) \bar{C}_{sr} \equiv a_k \]  

\[ (10) \]

Where \( \left[ \cdot \right] \) is integer function

The following equation satisfies probability density function \( P(x) \) of \( X(t) \):

\[ \frac{1}{2} \left( \frac{d^2}{dx^2} \right) \left\{ a(x)P(x) \right\} - \left( \frac{d}{dx} \right) \left\{ b(x)P(x) \right\} = -\lambda \pi_0 \delta(x-1) \quad (x > 0) \]  

\[ (11) \]

with boundary conditions

\[ \lim_{x \to 0} \left\{ \frac{1}{2} \left( \frac{d^2}{dx^2} \right) \left\{ a(x)P(x) \right\} - \left( \frac{d}{dx} \right) \left\{ b(x)P(x) \right\} \right\} = \lambda \pi_0 \]  

\[ (12) \]

\[ \lim_{x \to 0} P(x) = 0 \]  

\[ (13) \]

\[ \lim_{x \to \infty} P(x) = 0 \]  

\[ (14) \]

By Little's law, we have

\[ \bar{R}(r) = \sum_{s=1}^{r} \frac{\lambda_s R_s}{\Lambda_r} \]  

\[ (6) \]

The overall response -time of a \( r \) class, \( m \) -server system with FCFS service discipline or PR, arrival rate vector \( \bar{\lambda}_r = (\lambda_1, \lambda_2,\ldots,\lambda_r) \), square coefficient of variance vector of the inter-arrival times \( \bar{C}_{ar} = (C_{a1}, C_{a2},\ldots,C_{ar}) \), service rate vector \( \bar{\mu} = (\mu_1, \mu_2,\ldots,\mu_r) \) and service time square coefficient of variance vector \( \bar{C}_{sr} = (C_{s1}, C_{s2},\ldots,C_{sr}) \) will be denoted by \( R(d, \bar{\lambda}_r, \bar{C}_{ar}, \bar{\mu}_r, \bar{C}_{sr}, m) \) here \( d \) is either FCFS or PR.

The overall waiting time of the system with same parameters is similarly denoted by \( W(d, \bar{\lambda}_r, \bar{C}_{ar}, \bar{\mu}_r, \bar{C}_{sr}, m) \).

Let \( \{X(t): t \geq 0\} \) be the homogeneous diffusion processes corresponding to queue size \( \{N(t): t \geq 0\} \) and \( P(x) \) be the steady- state probability density function of the process \( X(t) \). Again suppose that \( \pi_k(k = 0,1,2,\ldots,N) \) denotes the steady state probability of the queueing process \( Q(t) \) and let \( b(x) \) and \( a(x) \) denote the infinitesimal mean and variance of the diffusion process.
where $\delta(.)$ is Dirac’s delta function and $\pi_0$ denotes the steady state probability mass at origin. $b(x)$ and $a(x)$ are piecewise continuous functions. Integrating (12) and using (13), we obtain

$$\frac{1}{2} \left( \frac{d}{dx} \right) \{a(x)P(x)\} - \{b(x)P(x)\} = \lambda \pi_0 \{1 - U(x - 1)\}$$

...(15)

we treat 0 as an elementary return boundary.

3. THE ANALYSIS

For $0 < x \leq 1$

$$P_1''(x) - r_1 P_1'(x) = 0$$

...(16)

$$P_1'(1) = \pi_0 \theta_1$$

...(17)

$$\lim_{x \to 0^+} \{P_1'(x) - r_1 P_1(x)\} = 0 \text{ where } r_1 = \frac{2b_1}{a_1}$$

...(18)

For $k - 1 < x \leq k \ (k = 2, 3, ..., m)$

$$P_k''(x) - r_k P_k'(x) = 0$$

...(19)

$$P_k'(k - \frac{1}{2}) = \pi_0 \theta_2$$

...(20)

$$P_k(k - 1 + 0) = P_{k-1}(k - 1) \quad \text{if} \quad \rho > (1 - r_m)^{-1}$$

...(21)

$$P_k(k) = P_{k+1}(k + 0) \quad \text{if} \quad \rho \leq (1 - r_m)^{-1}$$

...(22)

where $r_k = \frac{2b_k}{a_k}, \ (k = 2, 3, ..., m)$ and $\rho = \frac{\Lambda r}{m \mu_r}$

For $x > m$

$$P_{m+1}''(x) - r_m P_{m+1}'(x) = 0$$

...(23)

$$P_{m+1}(m) = -\pi_0 \theta_m r_m \frac{\rho}{(1 - \rho)}$$

...(24)

$$\lim_{x \to \infty} P_{m+1}(x) = 0$$

...(25)

where $\theta_k = \left( \frac{m \rho}{k!} \right)^k$ and $r_m \leq 0 \text{ when } \rho > 1$

The equations (16)-(18) have following solution in case of $0 < x \leq 1$

$$P_1(x) = \begin{cases} \pi_0 \theta_1 r_1 e^{r_1(x - 1)} & \text{if } b_1 \neq 0 \\ \pi_0 r_1 & \text{if } b_1 = 0 \end{cases}$$

...(26)

For $k - 1 < x \leq k \ (k = 2, 3, ..., m)$, the equations (19)-(22) yield

$$P_k(x) = \begin{cases} \pi_0 \{\theta_k + C_k (e^{r_k(x-k+1/2)} - 1)\} & \text{if } b_k \neq 0 \\ \pi_0 \theta_k r_k (2x - k) + P_{k-1}(k - 1)(x + k) & \text{if } b_k = 0 \end{cases}$$

...(27)

where $C_k = \begin{cases} \frac{\theta_k - P_{k-1}(k - 1)/\pi_0}{(1 - e^{-r_k/2})}, & \text{for } \rho > (1 - r_m)^{-1} \\ \frac{\theta_k - P_{k+1}(k + 0)/\pi_0}{(1 - e^{+r_k/2})}, & \text{for } \rho \leq (1 - r_m)^{-1} \end{cases}$

For $m < x < N$, the equations (23)-(25) give

$$P_{m+1}(x) = \pi_0 \theta_m r_m \frac{\rho}{1 - \rho} \left( 1 - e^{r_m(N-m)} \right)$$

...(28)

The steady state probabilities are obtained as

$$\pi_1 = \int_0^1 P_1(x) \, dx = \begin{cases} \pi_0 \theta_1 \left( \frac{e^{r_{1/2}} - e^{-r_{1/2}}}{r_1} \right), & \text{if } b_1 \neq 0 \\ \pi_0 \theta_1, & \text{if } b_1 = 0 \end{cases}$$

...(29)

$$\pi_k = \int_{k-1}^k P_k(x) \, dx$$

$$= \begin{cases} \pi_0 \theta_k + \frac{(\theta_k - P_{k-1} / \pi_0)}{(1 - e^{r_k/2})} \left( \frac{(e^{r_k/2} - e^{-r_k/2})}{r_k} \right), & \text{if } b_k \neq 0 \\ \pi_0 \theta_k(2k-2) + P_k(k-1)(4k-1), & \text{if } b_k = 0 \end{cases}$$

...(30)

$$\pi_N = \int_{k-1}^N P_{m+1}(x) \, dx = \pi_0 \theta_m r_m \frac{\rho}{1 - \rho} \left( 1 + \frac{r_m}{e^{r_m(N-m)} - 1} \right)$$

...(31)

$\pi_0$ is obtained by using the normalizing condition

$$\pi_0 + \int_0^N P(x) \, dx = 1$$

...(32)

$$\pi_0 = \left[ \frac{1}{1 - \sum_{k=2}^m P_{k-1}(k-1)} \left( 1 + \frac{e^{r_k/2} - e^{-r_k/2}}{r_k} \right) \right]$$

$$\left[ 1 + \sum_{k=1}^m \theta_k \left( e^{r_k/2} - e^{-r_k/2} \right) \left( 1/r_k + 1/r_k + e^{-r_{k+1}/2} \right) \right]$$

$$+ \frac{\rho}{1 - \rho} \left( 1 + \frac{r_m}{e^{r_m(N-m)} - 1} \right)^{-1}$$

...(33)

The mean number of customers including those which are in service is

$$L = \int_0^N xP(x) \, dx$$

...(34)

$$= \left[ \int_0^1 x \pi_0 r_1 e^{r_1(x-1/2)} \, dx + \int_1^2 x \pi_0 \left( \theta_k + C_k(e^{r_k(x-k+1/2)} - 1) \right) \, dx + \ldots \right]$$

$$+ \left. \int_0^N x \pi_0 \theta_m r_m \frac{\rho}{1 - \rho} \left( 1 - e^{r_m(x-N)} \right) \, dx \right|_m^n$$

\[
= \pi_0 \theta_1 e^{r_1 / 2} \left( \frac{r_1 - 1}{r_1} \right) + \frac{m}{\sum_{k=1}^{m} \pi_0 \theta_{k+1} \left( \frac{2k + 1}{2} \right) - \sum_{k=2}^{m} C_k}
+ \frac{m}{\sum_{k=2}^{m} C_k e^{-r_1 / 2} \left( \frac{r_1}{r_k} \right) \left( k e^{r_k} - (k - 1) e^{r_k} + 1 \right)}
+ \frac{\pi_0 \theta_m r_m \rho}{\left( 1 - \rho \right) \left( e^{-r_m \left( N - m \right)} - 1 \right)} \left[ N^2 - m^2 - 2 e^{2r_m N} \left\{ (N - m) r_m e^{r_m \left( m - N \right)} \right\} + (1 - e^{r_m \left( m - N \right)}) \right]
\]

The delay probability (DLY) and the mean queue length (QL) can be obtained as

\[
DLY = 1 - \sum_{k=0}^{m-1} \pi_k \quad \text{...(36)}
\]

\[
QL = \sum_{k=1}^{\infty} k \pi_{m+k} = \frac{\pi_0 \theta_m \rho \left( \rho \right)}{\left( 1 - e^{r_m} \right)} \quad \text{...(37)}
\]

**CONCLUSION**

The diffusion approximation model with reflecting boundaries at 0 and N for finite capacity G/G/m/N queue is developed to determine the queue size distribution by using the first two moments of inter-arrival time and service time each priority class job.

The approximate formula for performance indices may be helpful for decision makers to determine the delay characteristic for production system, computer system, communication networks, etc.

**REFERENCES**


