Higher Derivative Estimates for the 3D Navier-Stokes Equation

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Abstract:
In this paper, a non linear family of spaces, based on the energy dissipation is introduced. This family bridges an energy space (containing weak solutions to Navier-Stokes equation) to a critical space (invariant through the canonical scaling of the Navier-Stokes equation). This family is used to get uniform estimates on higher derivatives to solutions to the 3D Navier-Stokes equations. Those estimates are uniform, up to the possible blowing-up time. The proof uses blow-up techniques. Estimates can be obtained by this means thanks to Galilean invariance of the transport part of the equation.

Keywords: Navier-Stokes equation, fluid mechanics, blow-up techniques.

1. INTRODUCTION
Mechanics is an area of science concerned with the behaviour of physical bodies when subjected to forces or displacements, and the subsequent effects the bodies on their environment. Fluid mechanics is the branch of physics which involves the study of fluids (liquids, gases and plasmas) and the forces on them. Fluid mechanics can be divided into fluid statics and fluid dynamics. This paper uses a non linear spaces, based on the energy dissipation is introduced. This family bridges an energy space (containing weak solution to Navier-Stokes equation) to a critical space (invariant through the canonical scaling of the Navier-Stokes equation). This family is used to get uniform estimates on higher derivatives to solution to the 3D Navier-Stokes equations. Those estimates are uniform, up to the possible blowing-up time. We established ‘introduced the estimates can be obtained by this means thanks to Galilean invariance of the transport part of the equation’.

2. PRELIMINARIES

2.1 Definition
A fluid is defined as either a gas or a liquid. Fluid mechanics is the study of behavior of liquids and gases. More properly defined fluid mechanics is the study of fluids and forces on them.

2.2 Definition
Let \( \mathbf{v}(x, t) \) be a three dimensional vector field the velocity of the fluid and let \( p(x, t) \) be the pressure of fluid the Navier-Stokes equation

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f}(x, t)
\]

Where \( \nu > 0 \) is kinematic viscosity \( \mathbf{f}(x, t) \) the external force \( \nabla \) is gradient operator and \( \Delta \) is laplacian.

2.3 Definition
A fluid is incompressible if the density of a fluid particle is constant.

In other words
The rate of change of \( \rho(\mathbf{x}, t) \) following the particle path is zero

\[
\frac{\partial \rho}{\partial t} = 0
\]

The continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Reducts the equation

\[
\nabla \cdot \mathbf{u} = 0
\]

2.4 Definition
Let the random variable \( X \) have a distribution of probability about which we assume only that there is a finite variance \( \sigma^2 \). Then for every \( K > 0 \),

\[
P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}
\]

Or equivalently

\[
P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}
\]

This equation is known as Tchebyshev’s inequality.

3. HIGHER DERIVATIVES ESTIMATE FOR THE 3D NAVIERS-STOKES EQUATION

3.1 Theorem
For any \( 0 < \delta < 1 \), there exists \( \gamma > 0 \) and a constant \( C > 0 \) such that for any \( u \) solution to (1) (3), with \( u^0 \in L^2(\mathbb{R}^3) \), we have

\[
\int_0^\infty \int_{\mathbb{R}^3} \left( |M (\Delta)^{-\delta/2} \nabla^2 p| + |\nabla^2 p| + |\nabla u|^2 \right) dx dt \\
\leq C(\|u^0\|_{L^2(\mathbb{R}^3)} + \|u^0\|_{L^2(\mathbb{R}^3)}^{2(1+\gamma)})
\]

Moreover, \( \gamma \) converges to 0 when \( \delta \) converges to 0.
Proof

Integrating in \( x \) energy equation (2) gives that
\[
\int_{0}^{\infty} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt \leq \|u^0\|_{L^2_1(\mathbb{R}^3)}^2.
\] (5)

Together with
\[
\|u\|_{L^2_2(0,\infty;L^3(\mathbb{R}^3))} \leq \|u^0\|_{L^2(\mathbb{R}^3)}
\]

By sobolev imbedding and interpolation, this gives in particular that
\[
\|u\|_{L^2_2(0,\infty;L^3(\mathbb{R}^3))} \leq C\|u^0\|_{L^2(\mathbb{R}^3)}^2
\] (6)

For the pressure, we have \( \nabla^2 p \in L^1(\mathcal{H}). \) Indeed,
\[
\nabla^2 p = (\nabla^2 \Delta)^{-1} \sum_{i} \partial_i u_i \partial_j u_j
\]
\[
= (\nabla^2 \Delta)^{-1} \sum_{i} (\partial_i u_i) \nabla u_i.
\]

For any \( i \), we have \( \text{rot}(\nabla u_i) = 0 \) and \( \text{div } \partial_i u = 0 \). Hence, from the div-rot lemma, we have
\[
\|\sum_{i} \partial_i u_i \nabla u_i\|_{L^1(\mathcal{H})} \leq \|\nabla u\|_{L^2}^2.
\]

But \( \nabla^2 \Delta^{-1} \) is a Riesz operator (in \( x \) only) which is bounded from \( \mathcal{H} \) to \( \mathcal{H} \). Hence:
\[
\|\nabla^2 P\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \leq C \|\nabla^2 P\|_{L^1(\mathbb{R}^3 \times \mathcal{H}(\mathbb{R}^3))}
\]
\[
\leq C \|\nabla u\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2.
\] (7)

Using the sobolev imbedding with Hardy space, we get from the second estimate of (7) that for any \( 0 < s < 1 \),
\[
\|(-\Delta)^{-s/2} \nabla^2 p\|_{L^2_1(0,\infty;L^p(\mathbb{R}^3))} \leq C\|u^0\|_{L^2}^2
\] (8)

For
\[
\frac{1}{p} = 1 - \frac{s}{3}.
\]

We have also
\[
(-\Delta)^{-s/2} \nabla^2 p = \sum_{i} \left[(-\Delta)^{-3/2} \nabla^2 \partial_i\right] (\partial_i u_i, u_i).
\]

The operator \( (-\Delta)^{-3/2} \nabla^2 \partial_i \) are Riesz operators so, together with (5) (6), we have
\[
\|(-\Delta)^{-s/2} \nabla^2 p\|_{L^2_1(0,\infty;L^p(\mathbb{R}^3))} \leq C\|u^0\|_{L^2(\mathbb{R}^3)}^2.
\] (9)

By interpolation with (8), using theorem 2.1 with \( \theta = \frac{1}{1 + 4s} \), we find
\[
\|M\left[(-\Delta)^{-s/2} \nabla^2 p\right]\|_{L^{1+y}(0,\infty;L^3(\mathbb{R}^3))} \leq C\|u^0\|_{L^2(\mathbb{R}^3)}^2
\]

With
\[
\delta = \frac{5s}{1 + 4s}, \quad \gamma = \frac{s}{1 + 3s}
\]

Note that \( \gamma \) converges to 0 when \( \delta \) goes to 0.

This, together with (7) and (5), gives the result
\[
\int_{\mathbb{R}^3} \phi(x) \, dx = 1
\] (10)

For any \( \varepsilon > 0 \), we define
\[
u(s) = \int_{\mathbb{R}^3} \phi^{(\varepsilon y)\varepsilon y} \, dy.
\] (11)

Note that \( \nu \in L^\infty(0,\infty;C^\infty(\mathbb{R}^3)) \) and \( \text{div } u = 0 \). We define the flow:
\[
\frac{dX}{ds} = u(s, X(s, t, x))
\]
\[
X(t, t, x) = x.
\] (12)

Note that the flow \( X \) depends on \( \varepsilon \). Consider, for any \( \nu < \delta < 1 \) and \( \eta > 0 \)
\[
\frac{\Omega^\varepsilon}{\varepsilon} = \frac{\varepsilon}{1 - 4\varepsilon} \int_{-\varepsilon}^{1} \int_{B_2t} F^\varepsilon(s, X(s, t, x), y) \, ds \, dy \leq \eta^\varepsilon\varepsilon^\delta.
\]

Where
\[
F^\varepsilon(s, t, x) = \left| M((-\Delta)^{-s/2} \nabla^2 p) \right|^{1+y} + |\nabla u|^2 + |\nabla^2 p|,
\]

we define \( \delta \) and \( \eta \) in Theorem 2.5.

We then have the following Theorem.

4.BLOW UP METHOD ALONG THE PROJECTILE

4.1 Theorem

For any \( \gamma > 0 \) and any \( 0 < \delta < 1 \), there exists a constant \( \varepsilon < 1 \), and a sequence of constants \( \{C_n\} \) such that for any solution \( (u, P) \) of (1) (3) in \( Q_2 \) verifying
\[
\int_{\mathbb{R}^3} \phi(y) u(t, x) \, dx = 0, \quad t \geq -4,
\] (21)
\[
\int_{-4}^{0} \int_{B_2} \left| \nabla u \right|^2 \, dx \, dt \leq \tilde{\eta},
\] (22)
\[
\int_{-4}^{0} \int_{B_2} |\nabla^2 P| \, dx \, dt \leq \tilde{\eta},
\] (23)
\[
\int_{-4}^{0} \int_{B_2} \left| M\left[(-\Delta)^{-s/2} \nabla^2 p\right]\right|^{1+y} \, dx \, dt \leq \tilde{\eta},
\] (24)

The velocity \( u \) is infinitely differentiable in \( x \) at \( (0,0) \) and
\[
|\nabla u(0,0)| \leq C_n.
\]

Proof

We want to apply Theorem 2.4. Then, by a bootstrapping argument we will get uniform controls on higher derivatives.
For this, we first need a control of \( u \) in \( L^\infty(L^2) \) and a control on \( P \) in \( L^{r+1}(L^1) \). The equation is on \( \nabla P \) (not the pressure itself).

Therefore, changing \( P \) by \( P - \int_{\Omega_2} \psi P \, dx \) we can assume without loss of generality that

\[
\int_{\Omega^3} \psi(x) P(t,x) \, dx = 0, \quad -4 < t < 0.
\]

To get a control in \( L^{r+1}(L^1) \) on the pressure it is then enough to control \( \nabla P \).

**STEP 1: Control on \( u \) in \( L^\infty(L^{3/2}) \) in \( Q^{3/2}_1 \)**

In equation (27) there exists a constant \( c \), depending only on \( \psi \), such that for any 
\[
-4 < t < 0
\]

\[
\|u(t)\|_{L^\infty(\Omega_2)} \leq c \|\nabla u(t)\|_{L^2(\Omega_2)}.
\]  

(25)

So

\[
\|u(\nabla)u\|_{L^2(-4,0;L^2(\Omega_2))} \leq c \|\nabla u\|_{L^2(\Omega_2)} \leq c \eta.
\]

We need the same control on \( \nabla P \). First, multiplying (1) by \( \psi(x) \), integrating in \( x \), and Hypothesis (21), we find for any 
\[
-4 < t < 0
\]

\[
\int_{\Omega} \psi(\nabla)u(\nabla) ud\sigma + \int_{\Omega} \psi(\nabla) \nabla Pd\sigma - \int_{\Omega} \Delta u \psi ud\sigma = 0.
\]  

(26)

So

\[
\int_{\Omega} \psi(\nabla)\nabla Pd\sigma \leq c \left( \|\nabla u\|_{L^2(\Omega_2)}^2 + \|u\|_{L^2(-4,0;L^2(\Omega_2))} \right) \leq c \sqrt{\eta}.
\]

But, as for \( u \),

\[
\|\nabla P - \nabla \psi Pd\sigma\|_{L^2(-4,0;L^2(\Omega_2))} \leq c \|\nabla u\|_{L^2(\Omega_2)}^2.
\]

So, finally

\[
\|\nabla u\|_{L^2(-4,0;L^2(\Omega_2))} \leq c \sqrt{\eta}.
\]  

(27)

Note that

\[
\frac{3}{2} \frac{u}{|u|^{1/2}} \Delta u = \frac{3}{2} \frac{u}{|u|^{1/2}} \frac{\partial u}{\partial r} \frac{|u|^2}{2} = \frac{3}{2} \frac{u}{|u|^{1/2}} \frac{|u|^2}{2} = \frac{3}{2} \frac{|u|^2}{2} \frac{|u|^2}{|u|^{1/2}} = \frac{3}{2} \frac{|u|^2}{2} + \frac{3}{4} |\nabla u|^2.
\]

Since \( |\nabla u| \geq |\nabla u| \)

We consider \( \psi \in C^\infty(\mathbb{R}^4) \) a nonnegative function compactly supported in \( Q_1 \) with \( \psi = 1 \) in \( Q_1 \) and

\[
|\nabla_x \psi_1| + |\nabla^2_x \psi_1| \leq c.
\]

Multiplying (1) by \( \psi_1(t,x)u/|u|^{1/2} \) and integrating in \( x \) gives

\[
\frac{d}{dt} \int \psi_1(t,x) |u|^{3/2} dx
\]

\[
\leq \int (|\partial_x \psi_1| + |\Delta \psi_1|) |u|^{3/2} dx
\]

\[
+ \frac{3}{2} \left\| \psi_1^{1/2} |u|^{1/2} \right\|_{L^2(\Omega_2)} \left\| \psi_1^{1/2} (u \nabla u) \right\|_{L^2(\Omega_2)}
\]

\[
+ \| \nabla P \|_{L^{3/2}(\Omega_2)} \leq \int (|\partial_x \psi_1| + |\Delta \psi_1|) |u|^{3/2} dx
\]

\[
+ \frac{3}{2} \left( \int \psi_1(t,x) |u|^{3/2} dx \right)^{1/3} \left\| (u \nabla u) \right\|_{L^3(\Omega_2)}
\]

\[
+ \| \nabla P \|_{L^{3/2}(\Omega_2)} \leq \alpha(t) \left( 1 + \int \psi_1(t,x) |u|^{3/2} dx \right).
\]

With

\[
\alpha(t) = \int (|\partial_x \psi_1| + |\Delta \psi_1|) |u|^{3/2} dx
\]

\[
+ \frac{3}{2} \left\| (u \nabla u) \right\|_{L^2(\Omega_2)} + \| \nabla P \|_{L^{3/2}(\Omega_2)}
\]

From (25) and (26)

\[
\|u\|_{L^2(-4,0)} \leq c \sqrt{\eta}.
\]

Denoting \( Y(t) = \int \psi_1(t,x) |u|^{3/2} dx \), we have

\[
Y(t) \leq \alpha Y, \quad Y(-4) = 1.
\]

Gronwall's lemma gives that for any \(-4 < t < 0\) we have

\[
Y(t) \leq \exp \left( \int_{-4}^t \alpha(s) ds \right).
\]

Hence, for \( \eta \) small enough:

\[
\|u\|_{L^\infty(-3/2,0;L^{3/2}(\Omega_2))} \leq C \eta^{1/3}.
\]  

(28)

**Step 2: Control on \( u \) in \( L^\infty(L^2) \) in \( Q_1 \)**

We consider \( \psi \in C^\infty(\mathbb{R}^4) \) a nonnegative function compactly supported in \( Q_3/2 \) with \( \psi_2 = 1 \) in \( Q_3/2 \) and

\[
\|\nabla_x \psi_2\| + \|\nabla^2_x \psi_2\| \leq C.
\]

Multiplying inequality (3) by \( \psi_2 \) and integrating in \( x \) gives

\[
\frac{d}{dt} \int \psi_2 (\frac{|u|^2}{2}) \, dx
\]

\[
\leq \int u \nabla \psi_2 \left( \frac{|u|^2}{2} + p \right) \, dx + \int (\partial_t \psi_2 + \Delta \psi_2) \frac{|u|^2}{2} \, dx
\]

Equalities (25) together with (27) and sobolev imbedding gives

\[
\|u\|_{L^\infty(-3/2,0;L^2(\Omega_2))} \leq C \eta^{1/2}.
\]

Together with (28), this gives that

\[
|u|_{L^\infty(-1,0;L^2(\Omega_2))} \leq C \eta^{1/4}.
\]  

(29)

**Step 3: \( L^\infty \) bound in \( Q_2 \)**
We need now to get better integrability in time on the pressure.

From (26) and (29), we get
\[
\left\| \int \phi(x) \nabla P \, dx \right\|_{L^2(-1,0)} \leq C \sqrt{\bar{\eta}}
\]

With Theorem 2.1 and (24), this gives for \( \gamma < 1 \)
\[
\| \nabla P \|_{L^{1+\gamma}(-1,0; L^2(\Omega_t))} \leq C \sqrt{\bar{\eta}}
\]
Together with (29), (22), and Theorem 2.4, shows that for \( \bar{\eta} \) small enough,

we have
\[
|u| \leq 1 \quad \text{in } Q_{1/2}.
\]

**Step 4: Obtaining more regularity**

We now obtain higher derivatives estimate by a standard bootstrapping method.

We give the details carefully to ensure that the bounds obtained are universal, that is, do not depend on the actual solution \( u \).

For \( n \geq 1 \) we define \( r_n = 2^{-n-3}, \bar{r}_n = B_r \), and \( \bar{Q}_n = Q_{r_n} \).

We denote also \( \bar{\psi}_n \) such that \( 0 \leq \bar{\psi}_n \leq 1 \), \( \bar{\psi}_n \in C^{\infty}(\mathbb{R}^4) \),
\[
\bar{\psi}_n(t,x) = 1 \quad (t,x) \in \bar{Q}_n
\]
\[
= 0 \quad (t,x) \in \bar{Q}_{n-1}
\]

For any \( n \) we have
\[
\partial_t \nabla^2 + \partial_t A_n + \nabla \dot{R}_n - \Delta \nabla^2 u = 0. \quad (30)
\]

With
\[
A_n = \nabla^2(u \bar{\psi}_n), \quad R_n = \nabla \bar{\psi}_n
\]

So we have
\[
\| A_n \|_{L^p(\bar{Q}_{n-1})} \leq C_n \| u \|_{L^2(-r_n,0; W^{2,\infty}(\bar{Q}_{n-1}))} \quad (31)
\]

And from Theorem 2.3 we can split \( R_n \)
\[
R_n = R_{1,n} + R_{2,n},
\]

With
\[
\| R_{1,n} \|_{L^p(\bar{Q}_{n-1})} \leq C_n \| A_n \|_{L^p(\bar{Q}_{n-2})} \quad (32)
\]
\[
\| R_{1,n} \|_{L^1(-r_n,0; W^{2,\infty}(\bar{Q}_{n-1}))} \leq C_n \left( \| A_n \|_{L^p(\bar{Q}_{n-2})} + \| \nabla P \|_{L^2(\bar{Q}_{n-2})} \right)
\]
\[
\leq C_n \left( \| A_n \|_{L^p(\bar{Q}_{n-2})} + 1 \right) \quad (33)
\]

Moreover we have
\[
\partial_t (\bar{\psi}_n \nabla^2 u) - \Delta (\bar{\psi}_n \nabla^2 u) = -\partial_t (A_n \bar{\psi}_n) + \nabla \nabla R_n + 
\]
\[
\nabla (\bar{\psi}_n \nabla R_n) + \nabla (\bar{\psi}_n \nabla R_n) + \Delta (\bar{\psi}_n \nabla^2 u) - \nabla (\bar{\psi}_n \nabla R_n) + \nabla (\bar{\psi}_n \nabla R_n)
\]
\[
\Delta (\bar{\psi}_n \nabla^2 u) - 2 \partial_t (\bar{\psi}_n \nabla^2 u) + (\partial_t \bar{\psi}_n) \nabla^2 u
\]

Note that \( \bar{\psi}_n \nabla^2 u = 0 \) on \( \partial \bar{Q}_{n-1} \). So
\[
\bar{\psi}_n \nabla^2 u = V_{1,n} + V_{2,n}
\]

(34)

With
\[
\partial_t V_{1,n} - \Delta V_{1,n} = -\nabla (A_n \bar{\psi}_n) + \nabla \nabla R_n - \nabla (\bar{\psi}_n \nabla R_n)
\]
\[
\Delta \bar{\psi}_n \nabla^2 u - 2 \partial_t (\bar{\psi}_n \nabla^2 u) + (\partial_t \bar{\psi}_n) \nabla^2 u
\]
\[
= F_n,
\]

\[
V_{1,n} = 0 \quad \text{for } t = -r_{n-1}^2,
\]

And
\[
\partial_t V_{2,n} - \Delta V_{2,n} = -\nabla (\bar{\psi}_n \nabla R_n) + R_{2,n} (\nabla \bar{\psi}_n),
\]

\[
V_{2,n} = 0 \quad \text{for } t = -r_{n-1}^2
\]

From (30) and (31), we have
\[
\| F_n \|_{L^p(-r_{n-1},0; W^{1,p}(\bar{Q}_{n-1}))} \leq C_n \left( 1 + \| u \|_{L^2(-r_{n-1},0; W^{2,\infty}(\bar{Q}_{n-2}))} \right)
\]

So, from Theorem 2.2,
\[
\| V_{1,n} \|_{L^p(-r_{n-1},0; W^{1,p}(\bar{Q}_{n-1}))} \leq C \| F_n \|_{L^p(-r_{n-1},0; W^{1,p}(\bar{Q}_{n-1}))}
\]
\[
\| V_{2,n} \|_{L^1(-r_{n-1},0; W^{1,\infty}(\bar{Q}_{n-1}))} \leq C \left( \| \nabla \bar{\psi}_n \nabla V_{2,n} \|_{L^1(-r_{n-1},0; W^{1,\infty}(\bar{Q}_{n-1}))} \right)
\]

\[
\leq C_n \left( 1 + \| u \|_{L^2(-r_{n-1},0; W^{2,\infty}(\bar{Q}_{n-2}))} \right)
\]

where we have used (31) and (33).

Hence from (34) and using that \( \bar{\psi}_n = 1 \) on \( \bar{Q}_n \), we have
\[
\| \nabla u \|_{L^p(-r_{n-1},0; W^{1,p}(\bar{Q}_{n-1}))} \leq C_n \left( 1 + \| u \|_{L^2(-r_{n-1},0; W^{2,\infty}(\bar{Q}_{n-2}))} \right)
\]

By induction we find that for any \( n \geq 1 \), and any \( 1 \leq p < \infty \), there exists a constant \( C_{n,p} \) such that
\[
\| u \|_{L^2(-r_n,0; W^{2,\infty}(\bar{Q}_{n-1}))} \leq C_{n,p}
\]

This is true for any \( p \), so for \( n \) fixed, taking \( p \) big enough and using sololev imbedding, we show that for any \( 1 \leq q < \infty \), there exists a constant \( C_{n,q} \) such that
\[
\| u \|_{L^q(-r_n,0; W^{2,\infty}(\bar{Q}_{n-1}))} \leq C_{n,q}
\]

As (31), we get
\[
\| A_n \|_{L^p(-r_{n-1},0; W^{2,\infty}(\bar{Q}_{n-1}))} \leq C_n
\]

From Theorem 2.2 we get
\[ \| R_{1,n} \|_{L^1((-r^{2+n+4}_n;0;W^{1,\infty}(\mathbb{R}^{n+4}))} \leq C_n, \]
\[ \| R_{2,n} \|_{L^1((-r^{2+n+4}_n;0;W^{1,\infty}(\mathbb{R}^{n+4}))} \leq C_n. \]

Hence
\[ \| \partial_t \nabla^n u \|_{L^1((-r^{2+n+4}_n;0;W^{1,\infty}(\mathbb{R}^{n+4}))} \leq C_n, \]

And finally
\[ \| \nabla^n u \|_{L^\infty(\mathbb{R}^{n+4})} \leq C_n. \]

**CONCLUSION**

In this dissertation, we study the concept of partial regularity of solutions to Navier-Stokes equations in chapter IV. In chapter III blowing up method were investigated. In chapter II higher derivatives estimates of the 3D Navier-Stokes equation were discussed. The definitions which are essential for the above were given in chapter I.

**BIBLIOGRAPHY**


