Stability of Total Vertex Irregularity Strength of Helm Graphs

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Abstract:
The problem of finding irregularity strength of graphs was proposed by Chartrand [3] and has proved to be difficult in general. There are not many graphs for which irregularity strength is known. In this paper, we study the change in the total vertex irregularity strength of helm graph by adding an edge from its complement.

I. INTRODUCTION

Let \( G = (V, E) \) be a simple graph. By a total labeling of a graph we will mean an assignment \( f : E \cup V \rightarrow \mathbb{Z}^+ \) to the edges and vertices of G. The weight of a vertex \( v \in V \), is defined by \( w(v) = f(v) + \sum_{vu \in E} f(vu) \). Moreover, the weighting \( f \) is called irregular if for each pair of different vertices their weights are distinct. The total vertex irregularity strength, \( tvs(G) \), is the minimum value of the largest label over all such irregular assignments.

In [2], Martin Baca et al., determined the total vertex irregularity strength of complete graphs, prisms and star graphs. A Helm graph \( H_n \) is the graph obtained from a wheel by attaching a pendant edge at each vertex of the \( n \)-cycle.

The vertex set of \( H_n \) is \( V = \{u, v_i, u_i : 1 \leq i \leq n \} \) and the edge set of \( H_n \) is \( E = \{uv_i, v_i v_{i+1}, v_i u_i : 1 \leq i \leq n \} \), with indices taken modulo \( n \).

Example

**Helm graph \( H_5 \)**

A **generalised friendship graph** \( f_{m,n} \) is a collection of \( m \) cycles of order \( n \) meeting in a common vertex. The vertex set of \( f_{m,n} \) is \( V = \{v_i^j : 1 \leq i \leq n \} \) and the edge set of \( f_{m,n} \) is \( E = \{v_i^j v_{i+1}^j : 1 \leq i \leq m \cap 0 \leq j \leq n-1 \} \), with indices taken modulo \( n \).

Example

**Friendship graph \( f_{3,6} \)**

A **flower graph** \( F_n \) is the graph obtained from a helm by joining each pendant vertex to the helm. The vertex set of \( F_n \) is \( V = \{v, v_i, u_i : 1 \leq i \leq n \} \) and the edge set of \( F_n \) is \( E = \{vv_i, vu_i, v_i u_{i+1}, v_i u_i : 1 \leq i \leq n \} \), with indices taken modulo \( n \).

Example

**Flower Graph \( F_5 \)**

A **web graph** \( W_{b,n} \) is the graph obtained from a helm by joining the pendant vertices to form an \( n \)-cycle. The vertex set of \( W_{b,n} \) is \( V = \{v, v_j, u_j : 1 \leq j \leq n \} \) and the edge set of \( W_{b,n} \) is \( E = \{vu_i, v_j u_i, v_i v_{j+1}, v_i^2 u_i : 1 \leq j \leq n \} \), with indices taken modulo \( n \).
Example

Web Graph \( W_{b_5} \)

Theorem

For \( n \geq 4 \), the total vertex irregularity strength of \( H_n \) is

\[
tvs(H_n) = \left\lceil \frac{n+1}{2} \right\rceil
\]

Proof.

The vertex set and edge set of the helm \( H_n \) are

\[
V(H_n) = \{ u_i, v_i : 1 \leq i \leq n \} \cup \{ u \}
\]

\[
E(H_n) = \{ v_i v_{i+1} ; u_i v_i ; uv_i : 1 \leq i \leq n \}
\]

Consider the vertices of degree 1. There are \( n \) such vertices, and if we want to use only the labels \( 1, 2, \ldots, s \), the lowest and highest weights that we can obtain are respectively 2 and 2, which implies that \( 2s - 2 + 1 \geq n \) and thus

\[
tvs(H_n) \geq \left\lceil \frac{n+1}{2} \right\rceil
\]

To show that \( tvs(H_n) \leq \left\lceil \frac{n+1}{2} \right\rceil \), we define a labeling \( \emptyset : \)

\[
V(H_n) \cup E(H_n) \to \{ 1, 2, \ldots, \left\lceil \frac{n+1}{2} \right\rceil \}
\]

as follows:

\[
\emptyset (v_i v_{i+1}) = \emptyset (u) = \emptyset (u v_i) = \left\lceil \frac{n+1}{2} \right\rceil, \text{ for } 1 \leq i \leq n
\]

\[
\emptyset (u) = \emptyset (u_i) = \left\{ \begin{array}{ll}
1, & \text{for } i = 1 \\
2, & \text{for } 2 \leq i \leq 4 \\
\left\lceil \frac{n+1}{2} \right\rceil, & \text{for } 5 \leq i \leq n
\end{array} \right.
\]

\[
\emptyset (v_i u_i) = \left\{ \begin{array}{ll}
1, & \text{for } i = 1, 2 \\
2, & \text{for } i = 3 \\
\left\lceil \frac{n+1}{2} \right\rceil, & \text{for } 4 \leq i \leq n
\end{array} \right.
\]

This labeling gives the weight of the vertices, \( u, u_i \) and \( v_i \) for \( 1 \leq i \leq n \), as follows:

\[
\omega(t) = \left\lceil \frac{n+1}{2} \right\rceil (n+1);
\]

\[
\omega(t) = i + 1;
\]

\[
\omega(t) = 3 \left\lceil \frac{n+1}{2} \right\rceil + 1 + i.
\]

The weight of the vertices are distinct, and thus \( \emptyset \) is the vertex irregular total labeling of the helm graph \( H_n \).

Example

Let \( e \) be any edge of \( G \), then \( e \) is called a total positive edge of \( G \), if \( tvs(G + e) > tvs(G) \). The total negative edge and total stable edge of \( G \) if \( tvs(G + e) < tvs(G) \) and \( tvs(G + e) = tvs(G) \) respectively. If joining of any two non-adjacent vertices of \( G \), by an edge increases the total vertex irregularity strength of \( G \), then \( G \) is a total positive graph. If it decreases the total vertex irregularity strength of \( G \), then \( G \) is a total negative graph.

II. MAIN RESULT

In this section, we study stability of total vertex irregularity strength of helm graphs \( H_n \) for \( n \equiv 0 \mod 4 \) and \( n \equiv 1 \mod 4 \).

Theorem

For \( n \equiv 0 \mod 4 \), the Helm graph \( n \geq 4 \) is a total negative graph.

Proof.

The vertex set and edge set of \( H_n \) are

\[
V(H_n) = \{ u_i, v_i : 1 \leq i \leq n \} \cup \{ u \}
\]

\[
E(H_n) = \{ v_i v_{i+1}, u_i v_i, uv_i, u_{n-1}u_n : 1 \leq i \leq n \}
\]

Add the edge \( u_{n-1}u_n \) to \( H_n \), then define the total labeling, \( \varphi \) as follows:

(i) \( \varphi (u) = 1 \)

(ii) \( \varphi (v_i) = \left\lceil \frac{n+1}{2} \right\rceil, 2 \leq i \leq n \)

(iii) \( \varphi (u) = \left\lceil \frac{n+1}{2} \right\rceil - \frac{n}{2}, 2 \leq i \leq n - 2 \)

(iv) \( \varphi (u_i) = \left\lceil \frac{n+1}{2} \right\rceil, 2 \leq i \leq n \)

(v) \( \varphi (v_i u_i) = \left\lceil \frac{n+1}{2} \right\rceil, 2 \leq i \leq n \)

(vi) \( \varphi (v_i v_{i+1}) = \left\lceil \frac{n+1}{2} \right\rceil, i \text{ is odd}; \left\lceil \frac{n+1}{2} \right\rceil, i \text{ is even}; \frac{n+1}{2} \leq i \leq n \)

(vii) \( \varphi (u_{n-1}u_n) = 1 \)

By the above irregularity total labeling we get,

\[
tvs(H_n + u_{n-1}u_n) \leq \left\lceil \frac{n+1}{2} \right\rceil \leq tvs(H_n)
\]

Thus, \( u_{n-1}u_n \) is a total negative edge. Since \( H_n + u_i u_{i+1} \equiv H_n + u_{n-1}u_n, 1 \leq i \leq n, u_i u_{i+1} \) is a total negative edge for all \( i \).

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Example

\[ tvs(H_9 + u_7u_9) = 4 \]

Theorem

For \( n \equiv 1(mod\ 4) \), the Helm graph \( H_n, n \geq 4 \) is a total negative graph.

Proof.

Let the vertex set and edge set of \( H_n \) be

\[ V(H_n) = \{u,v; 1 \leq i \leq n\} \cup \{u\} \]

\[ E(H_n) = \{v_iv_{i+1}, uv_i, uu_{n-1}u_n; 1 \leq i \leq n\} \]

Add the edge \( u_{n-1}u_n \) to \( H_n \), and then define the total labeling, \( \varphi \) as follows:

(i) \( \varphi(u) = 1 \)

(ii) \( \varphi(v_i) = \begin{cases} \left\lceil \frac{n+5}{4} \right\rceil; & i = 1 \\ \left\lceil \frac{n}{2} \right\rceil; & 2 \leq i \leq n \end{cases} \)

(iii) \( \varphi(u_i) = \begin{cases} i; & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, i \neq n - 1 \\ \left\lfloor \frac{n}{2} \right\rfloor; & i = n - 1 \end{cases} \)

(iv) \( \varphi(u_iv_i) = \begin{cases} 1; & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ i + 1 - \left\lfloor \frac{n}{2} \right\rfloor; & \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n - 3 \\ \frac{n-5}{4} + i; & i \text{ is even} \\ \frac{n-5}{4} + \frac{i+1}{2}; & i \text{ is odd} \end{cases} \)

(v) \( \varphi(v_iv_{i+1}) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 1; & \frac{n}{2} \leq i \leq n - 2 \\ \left\lfloor \frac{n}{2} \right\rfloor; & i = n - 1, n \end{cases} \)

(vi) \( \varphi(u_iv_{i+1}) = \frac{n}{2}; \forall i \)

By the above irregularity total labeling we get,

\[ tvs(H_n + u_{n-1}u_n) \leq \frac{n}{2} < tvs(H_n) \]

Thus, \( u_{n-1}u_n \) is a total negative edge. Since \( H_n + u_iu_{i+1} \equiv H_n + \)

Example

\[ tvs(H_9 + u_9u_9) = 4 \]

III. CONCLUSION

In this paper we proved that joining of any two consecutive pendant vertices in helm graph \( (H_n) n \geq 4 \), is a total negative edge for \( n \equiv 0(mod\ 4) \) and \( n \equiv 1(mod\ 4) \). Thus helm graph \( (H_n) n \geq 4 \), is a total negative graph for \( n \equiv 0(mod\ 4) \) and \( n \equiv 1(mod\ 4) \).

IV. REFERENCE

