Numerical Solutions of One Dimensional Ground Water Recharge through Unsaturated Porous Media with Parabolic Permeability

Priyanka S. Patel¹, K.A. Patel², Rtd. Prof. P.H. Bhathawala³
Research scholar, Pacific university, Udaipur, India¹
Department of Mathematics, Shri U.P. Arts, Smt. M.G. Panchal Science & Shri V.L. Shah Commerce College, Pilvai, India²
Department of Mathematics, VNSGU, Surat, India³

Abstract:
The present paper contains the numerical solution of one dimensional flow through unsaturated porous media. Here we discussed one-dimensional ground water recharge through unsaturated porous media with parabolic permeability, which are described by non-linear partial differential equation. This equation is solved by reduced differential transform method (RDTM). Numerical solution of the problem obtains through MATLAB.

Key words: Ground water recharge, parabolic permeability, unsaturated porous media, RDTM

1. INTRODUCTION

The unsteady and unsaturated flow of water through soil is due to content changes as a function of time and entire pore spaces are not completely fill with flowing liquid respectively. The ground water flow plays an important role in various fields like Agriculture, fluid dynamics, chemical engineering, Environmental problem, Bio-mathematic and nuclear waste disposal problems. The water infiltration system and underground disposal of seepage and waste water are encountered these flows. In, the present paper we have derived the mathematical model that confirm the hydrological situation of one dimensional vertical groundwater recharge by spreading. This physical phenomenon has confirmed by verma [1]. Such flow is of enormous importance in water resource science, soil engineering and agricultural sciences. Many researcher discussed this phenomenon from different perspective, such as klute and Hank Bower use a finite difference method, Philips utilize a transformation of variable technique, Mehta discussed multiple scale method, Verma has obtained Laplace transformation and similarity solution [1]. In recent decades, there has been great development in the numerical analysis and exact method solving partial differential equation for example, Adomain decomposition method, Homotopy perturbation method, parameter expanding method, Successive over Relaxation (S.O.R.,) method [5], Elzaki transform method[6], Crank-Nikolson finite difference method [7], Variational iteration method [8,10] and Differential Quadrature Method [9] and. Here we use Reduced differential transform method (RDTM) is very straightforward and effectual to get exact & numerical solution of the one dimensional ground water recharge problem with parabolic permeability.

2. Statement of the Problem:

In the investigated mathematical model, we consider that the groundwater recharge takes place over a large basin of such geological location that sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case, take flow is assumed vertically downloads through unsaturated porous media. It is assumed that diffusivity coefficient is equivalent to its average value over the whole range of moisture content, and the permeability of the media is continuous linear function of the problem yields a nonlinear partial differential equation for the moisture content.

3. Formulation of the Problem:

Following Klute [2].

We may write fundamental equation as below. The equation of continuity for unsaturated medium is given by

\[ \frac{\partial}{\partial t} (\rho_s \theta) = - \nabla M \]  \hspace{1cm} (1)

Where \( \rho_s \) is bulk density of the medium, \( \theta \) is the moisture content on a dry weight basis, and \( M \) is the mass flux of moisture.

From Darcy’s law for the motion of water in a porous media we get,

\[ V = -k \nabla \phi \]  \hspace{1cm} (2)

Where \( \nabla \phi \) represents the gradient of whole moisture potential, \( V \) the volume flux of moisture potential, and \( k \) the coefficient of aqueous conductivity. Combining equation (1) & (2) we obtain,

\[ \frac{\partial}{\partial t} (\rho_s \theta) = -\nabla(\rho k \nabla \phi) \]  \hspace{1cm} (3)

Where \( \rho \) is the fluid density. Since in the present case we consider that the flow takes place only in the vertical direction, equation (3) reduces to,

\[ \rho_s \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( k \frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} (\rho g \phi) \]  \hspace{1cm} (4)

Where \( \phi \) is the capillary pressure potential, \( g \) is the gravitational constant and \( \phi = \Psi - gz \).
The Positive direction of the z-axis is the same as that of the gravity.

Considering θ to be conducted by a single valued function, we may write (4) as,
\[
\frac{\partial \theta}{\partial t} = \frac{D}{\partial z} \left( \frac{\partial \theta}{\partial z} \right) - \frac{\rho}{\rho_s} \frac{\partial \theta}{\partial z} \tag{5}
\]

Where \( D = \frac{\rho}{\rho_s} k^2 \frac{\partial \Psi}{\partial \xi} \) and is called diffusivity coefficient.

Replacing \( D \) by its average value \( D_a \) and assuming \( k = k_o \theta, k_o = 0.232 \), we have
\[
\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_o \frac{\partial \theta}{\partial z} \tag{6}
\]

Considering water table to be situated at a depth \( L \) and putting:
\[
\frac{z}{L} = \frac{\xi}{L}, \quad \frac{D_a}{L^2} = T, \quad \beta_o = \frac{\rho}{\rho_s} k_o \frac{\partial \theta}{\partial z}
\]

One-dimensional Groundwater recharge through porous media with linear permeability is,
\[
\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \beta_o \frac{\partial \theta}{\partial z}
\]

Where \( \xi \) is penetration depth (dimensionless)

\( T = \) time (dimensionless)

\( \beta_o = \) flow parameter (cm²)

Set of appropriate boundary conditions are \( \theta(0,T) = \theta_o, \theta(1,T) = 1, \theta(\xi,0) = 0 \) \[1\]

But here we assumed, the permeability of the moisture content to have a parabolic distribution,

i.e., \( K = K_o + K_0 \theta^2 \) \( (K_o = 0.232) \) where \( k_i \) and \( k_o \) are constants. Then equation (5) becomes,
\[
\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_o \frac{\partial \theta}{\partial z}
\]

Considering the water table to be situated at a depth \( L \) and putting:
\[
\frac{z}{L} = \frac{\xi}{L}, \quad \frac{D_a}{L^2} = T, \quad \beta_o = \frac{\rho}{\rho_s} k_o \frac{\partial \theta}{\partial z}
\]

We may write the problem as below from the boundary value
\[
\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta_o \frac{\partial \theta}{\partial \xi} \tag{7}
\]

It may be mentioned for definiteness that a set of appropriate boundary conditions are
\[
\theta(0,T) = \theta_o, \quad \frac{\partial \theta}{\partial \xi}(1,T) = 0, \theta(\xi,0) = 0
\]

Where the moisture content throughout the region is zero initially, at the layer \( z=0 \) it is \( \theta_o \), and at the water table \( (Z=L) \) it is assumed to remain 100% throughout the process of investigation. It may be remarked that the effect of capillary action at the stationary groundwater level, being small is neglected.

We can write equation (7) as \( \theta_T = \theta_{\xi} - \beta_o \theta_{\xi} \)

The problem is solved by using Reduced Differential Transform method. The numerical values are shown by table. Curves indicate the moisture content corresponding to various time periods.

4. SOLUTION USING RDTM METHOD

The Basic definition of RDTM is given below

If the function \( u(x,t) \) is analytic and differential continuously with respect to time \( t \) and space \( x \) in the domain of interest then let
\[
U_k = \frac{1}{k!} \left[ \frac{\partial^k}{\partial t^k} u(x,t) \right]
\]

Where the t-dimensional spectrum function \( U_k(x) \) is the transformed function, \( u(x,t) \) represent transformed function. The differential inverse transform of \( U_k(x) \) is defined as follow
\[
u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{\partial^k}{\partial t^k} u(x,t) \right] t^k
\]

Apply RDTM on (7)
\[
(k+1)\theta_{(k+1)}(\xi) = \left[ (\theta_0)(\xi) \right] - \beta_o \sum_{r=0}^{k} \left[ \theta_r(\xi) \theta_{(k-r)}(\xi) \right] \tag{9}
\]

Now let \( \beta_o = 1 \) then put initial condition (8) into eq. (9). So we get \( \theta_{\xi}(\xi) \) as following

(1) $\theta_1(\xi) = \left[ (\theta_0)_{\xi} \right] - \left[ (\theta_0(\theta_0)_{\xi} \right]$
\[ \theta_1(\xi) = \left( \frac{e^\xi}{e-1} \right) - \left( \frac{e^\xi-1}{(e-1)^2} \right) \]
For second iteration let $k = 1$

\[ (1 + 1)\theta_{2(1)}(\xi) = \left[ (\theta_2)_{\xi} \right] - \sum_{r=0}^{1} (\theta_2(\theta_{2})_{\xi}) \]
\[ 2\theta_2(\xi) = \left( \frac{e^{\xi+1} - 4e^{\xi}}{(e-1)^2} \right) \]
\[ - \left( \frac{e^\xi-1}{(e-1)^2} \right) \]
\[ + \left( \frac{e^{\xi+1} - 2e^{2\xi}}{(e-1)^2} \right) \]
\[ \theta_2(\xi) = \frac{1}{2} \left( \frac{e^{\xi+2} - 6e^{2\xi+1} + 3e^{\xi+1} + 2e^{2\xi}}{(e-1)^3} \right) \]
For third iteration let $k = 2$

\[ (2 + 1)\theta_{2(2)}(\xi) = \left[ (\theta_2)_{\xi} \right] - \sum_{r=0}^{2} (\theta_2(\theta_{2})_{\xi}) \]
\[ 3\theta_2(\xi) = \left( \frac{e^{\xi+2} - 29e^{2\xi+1} + 54e^{\xi+1} + 20e^{\xi+1} - 29e^{\xi} - 29e^{\xi} - 16e^{4\xi}}{(e-1)^4} \right) \]
Thus we can generated other polynomials by proceeding in same way

Now by inverse transform.
\[ u(x, t) = \sum_{k=0}^{\infty} U_k(x)t^k \]
\[ \theta_0(\xi, T) = \theta_0(\xi)T^0 + \theta_1(\xi)T^1 + \theta_2(\xi)T^2 + \theta_3(\xi)T^3 ... \]
\[ \theta(\xi, T) = \frac{e^\xi - 1}{(e-1)^2} \]
\[ + \left( \frac{e^{2\xi} - 29e^{2\xi+1} + 54e^{\xi+1} + 20e^{\xi+1} - 29e^{\xi} - 16e^{4\xi}}{(e-1)^4} \right) \]

5. TABLE AND FIGURE
The following table shows the approximate moisture content of liquid for different values of $x$ at different time using RDTM

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$T=0$</th>
<th>$T=0.1$</th>
<th>$T=0.2$</th>
<th>$T=0.3$</th>
<th>$T=0.4$</th>
<th>$T=0.5$</th>
<th>$T=0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0612</td>
<td>0.0611</td>
<td>0.0611</td>
<td>0.0609</td>
<td>0.0607</td>
<td>0.0602</td>
<td>0.0593</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1289</td>
<td>0.1286</td>
<td>0.1282</td>
<td>0.1272</td>
<td>0.1257</td>
<td>0.1228</td>
<td>0.1166</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2036</td>
<td>0.2031</td>
<td>0.202</td>
<td>0.1995</td>
<td>0.1955</td>
<td>0.1881</td>
<td>0.1726</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2862</td>
<td>0.2854</td>
<td>0.2831</td>
<td>0.2786</td>
<td>0.2711</td>
<td>0.2574</td>
<td>0.2285</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3775</td>
<td>0.3763</td>
<td>0.3727</td>
<td>0.3657</td>
<td>0.3539</td>
<td>0.3325</td>
<td>0.2874</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4785</td>
<td>0.4768</td>
<td>0.4718</td>
<td>0.4623</td>
<td>0.4461</td>
<td>0.4168</td>
<td>0.3549</td>
</tr>
<tr>
<td>0.7</td>
<td>0.59</td>
<td>0.5879</td>
<td>0.5819</td>
<td>0.5704</td>
<td>0.5508</td>
<td>0.5153</td>
<td>0.4404</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7132</td>
<td>0.7111</td>
<td>0.7049</td>
<td>0.693</td>
<td>0.6728</td>
<td>0.6362</td>
<td>0.559</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8495</td>
<td>0.8479</td>
<td>0.8432</td>
<td>0.8343</td>
<td>0.8191</td>
<td>0.7917</td>
<td>0.7338</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
CONCLUSION

We conclude from the graph that as time increases, the moisture content also increases at each point in the basin and after sometime, it become constant. Also, at particular time, optimum moisture content rises with increase in length.

Application:

For agriculture purpose [4], the continued presence of water in excess of that needed for vegetation is harmful. Because of the ground water recharge the salinity of the soil can be reduced because of the increase of moisture content, due to the increase in moisture content the fertility of soil increases which helps the farmer in growing up a qualitative crop and in this case production of the crop and this case production of the crop will also increase and quality of the ground water also increase.

REFERENCES


