On Nano Generalized\(^{\text{a}}\) - Continuous and Irresolute Functions in Nano Topological Spaces

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Abstract:
The purpose of this paper is to introduce a new class of continuous and irresolute functions called nano generalized\(^{\text{a}}\)-continuous functions and nano generalized\(^{\text{a}}\)-irresolute functions. And to discuss some of its properties in terms of Ng\(^{\text{a}}\)-closed sets, Ng\(^{\text{a}}\)-closure, Ng\(^{\text{a}}\)-interior.

Keywords: Nano topology, Ng\(^{\text{a}}\)-closed sets, Ng\(^{\text{a}}\)-closure, Ng\(^{\text{a}}\)-interior, Nano g\(^{\text{a}}\)-continuous function, Nano g\(^{\text{a}}\)-irresolute function.

1. INTRODUCTION

In 1991, Balachandran [1] et.al, introduced and studied the notations of generalized continuous functions. Different types of generalizations of continuous functions were studied by various authors in the recent development of topology. Continuous function is one of the main functions in topology. Lellis Thivagar [5] introduced Nano topological space with respect to a subset X of a universe which is defined in terms of lower and upper approximations of X. The elements of Nano topological space are called Nano open sets. He has also defined Nano closed sets, Nano-interior and Nano closure of a set. He also introduced the weak forms of Nano open sets namely Nano-\(\alpha\) open sets, Nano semi open sets and Nano preopen sets. He also defined Nano continuous functions, Nano open mapping, Nano closed mapping and Nano Homeomorphism. In this paper some properties of generalized\(^{\text{a}}\)-continuous functions, Nano generalized\(^{\text{a}}\)-irresolute functions in Nano topological spaces are studied.

2. PRELIMINARIES

**Definition 2.1:** [6] A g\(^{\text{a}}\)-closed set\([20]\) if cl(A)\(\subseteq\)G whenever A\(\subseteq\)G and G is semi-open in \((X,\tau)\). The complement of a g\(^{\text{a}}\)-closed set is called a g\(^{\text{a}}\)-open set.

**Definition 2.2:** [6] A function \(f : (X,\tau) \rightarrow (Y,\sigma)\) is called g-continuous[4] if \(f^{-1}(V)\) is g-closed in \((X,\tau)\) for every closed set of \((Y,\sigma)\).

**Definition 2.3:** [2]

Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair \((U, R)\) is said to be the approximation space Let U be a non-empty finite set of objects called the universe and R be an equivalence relation. Let \(X \subseteq U\).

(1).The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by \(L_\text{R}(X)\)

That is, \(L_\text{R}(X) = \bigcup_{x \in X} \{R(x) : R(x) \subseteq X\}\) where \(R(X)\) denotes the equivalence class determined by \(x\).

(2). The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by \(U_\text{R}(X)\). That is, \(U_\text{R}(X) = \bigcup_{x \in X} \{R(x) : R(x) \cap X \neq \emptyset\}\)

(3). The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by \(B_\text{R}(X)\). That is, \(B_\text{R}(X) = U_\text{R}(X) - L_\text{R}(X)\)

**Property 2.4:** [2] If \((U, R)\) is an approximation space and \(X, Y \subseteq U\), then

(i) \(L_\text{R}(X) \subseteq X \subseteq U_\text{R}(X)\)

(ii) \(L_\text{R}(\emptyset) = U_\text{R}(\emptyset)\) and \(L_\text{R}(U) = U_\text{R}(U) = U\)

(iii) \(U_\text{R}(X \cup Y) = U_\text{R}(X) \cup U_\text{R}(Y)\)

(iv) \(U_\text{R}(X \cap Y) \subseteq U_\text{R}(X) \cap U_\text{R}(Y)\)

(v) \(L_\text{R}(X \cap Y) = L_\text{R}(X) \cap L_\text{R}(Y)\)

(vi) \(L_\text{R}(X \cup Y) = L_\text{R}(X) \cup L_\text{R}(Y)\)

(vii) \(L_\text{R}(X) \subseteq L_\text{R}(Y)\) and \(U_\text{R}(X) \subseteq U_\text{R}(Y)\) whenever \(X \subseteq Y\)

(viii) \(U_\text{R}(X^\circ) = \{L_\text{R}(X)^\circ\}\) and \(L_\text{R}(X^\circ) = \{U_\text{R}(X)^\circ\}\)

(ix) \(U_\text{R}U_\text{R}(X) = L_\text{R}U_\text{R}(X) = U_\text{R}(X)\)

(x) \(L_\text{R}L_\text{R}(X) = U_\text{R}L_\text{R}(X) = L_\text{R}(X)\)

**Definition 2.5:**[2]

Let U be the universe, R be an equivalence relation on U and \(\tau_R(X) = \{U, \emptyset, L_\text{R}(X), U_\text{R}(X), B_\text{R}(X)\}\) where \(X \subseteq U\). Then by property 2.3 \(\tau (X)\) satisfies the following axioms:

(i) \(U \emptyset \in \tau_R(X)\)

(ii) The union of the elements of any subcollection of \(\tau_R(X)\) is in \(\tau_R(X)\)

(iii) The intersection of the elements of any finite subcollection of \(\tau_R(X)\) is in \(\tau_R(X)\).

That is, \(\tau_R(X)\) is a topology on U called the Nanotopology on U with respect to X.
We call \((U, \tau_R(X))\) as the Nanotopological space. The elements of \(\tau_R(X)\) are called as Nano-open sets.

**Remark 2.6 [2]**

If \(\tau_R(X)\) is the Nanotopology on \(U\) with respect to \(X\), then the set \(B = \{U, \emptyset, L_R(U), U_R(U), B_R(U)\}\) is the basis for \(rR(X)\).

**Definition 2.7:** [2]

If \((U, \tau_R(X))\) is a Nano topological space with respect to \(X\) where \(X \subseteq U\) and if \(A \subseteq U\), then the Nano interior of \(A\) is defined as the union of all Nano-open subsets of \(A\) and it is denoted by \(NInt(A)\). That is, \(NInt(A)\) is the largest Nano-open subset of \(A\).

The Nano closure of \(A\) is defined as the intersection of all Nano-open sets containing \(A\) and it is denoted by \(Ncl(A)\). That is, \(Ncl(A)\) is the smallest Nano-closed set containing \(A\).

**Definition 2.8:**

Let \((U, \tau_R(X))\) be a nano topological space. A subset \(A\) of \((U, \tau_R(X))\) is called Nano generalized\(^{\sim}\)-closed set (briefly Ng\(^{\sim}\)-closed) if \(Ncl(A) \subseteq V\) where \(A \subseteq V\) and \(V\) is Nano Semi-Open.

**Definition 2.9:** [3]

Let \((U, \tau_R(X))\) and \((V, \tau_R(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano continuous on \(U\) if the inverse image of every Nano open set in \(V\) is Nano open in \(U\).

**Definition 2.10:** [4]

Let \((U, \tau_R(X))\) and \((V, \tau_R(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano \(g^1\)-continuous on \(U\) if the inverse image of every Nano open set in \(V\) is Nano \(g^1\)-open in \(U\).

**Definition 2.11:** [6]

A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is called Irresolute[6] if \(f^{-1}(V)\) is semi-open in \((X, \tau)\) for every semi-open set \(V\) of \((Y, \sigma)\).

**3. NANO GENERALIZED\(^{\sim}\)-CONTINUOUS FUNCTIONS AND IRRESOLUTE FUNCTIONS**

**Definition 4.1:**

Let \((U, \tau_R(X))\) and \((V, \tau_R(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano \(g^1\)-continuous on \(U\) if the inverse image of every Nano open set in \(V\) is Nano \(g^1\)-open in \(U\).

**Example 4.2:**

Let \(U = \{a, b, c, d\}\) with \(U/R = \{\{c\}, \{d\}, \{a, b\}\}\) and \(X = \{a, c\}\).

Then \(\tau_R(X) = \{U, \Phi, \{c\}, \{a, b, c\}, \{a, b\}\}\) which are open sets.

The Nano closed sets = \(\{U, \Phi, \{c\}, \{a, b, c\}, \{a, b\}\}\).

The Nano generalized\(^{\sim}\)-closed sets are \(\{U, \Phi, \{a, b, d\}, \{a, b, c\}, \{a, b, d\}\}\).

Nano \(g^1\)-open sets are \(\{\Phi, U, \{a, b, c\}, \{a, b, c\}, \{a, c\}, \{a, b\}\}\).

Let \(V = \{x, y, z, w\}\) with \(V/R = \{\{x, y\}, \{z\}, \{w\}\}\) and \(Y = \{x, y\}\) Then \(\tau_R(Y) = \{\{x, y\}, \{z\}, \{x, y\}\}\) which are open sets.

Define \(f: U \rightarrow V\) as \(f(a) = x, f(b) = y, f(c) = z, f(d) = w\).

Then \(f^{-1}(x) = \{a, b, c\}, f^{-1}(y, z) = \{a, b\}\) and \(f^{-1}(w) = U\).

That is the inverse image of every Nano open set in \(V\) is Nano \(g^1\)-open in \(U\).

Therefore \(f\) is Nano \(g^1\)-continuous.

**Definition 4.3:**

For every set \(A \subseteq U\), we define the Ng\(^{\sim}\)-closure of \(A\) to be the intersection of all Ng\(^{\sim}\)-closed sets containing \(A\).

In symbols \(Ng^{\sim} - cl(A) = \cap \{B: B\text{ is Nano generalized}^{\sim}\text{-closed set and } A \subseteq B\}\).

**Definition 4.4:**

For every set \(A \subseteq U\), we define the Ng\(^{\sim}\)-Closure of \(A\) to be the union of all Ng\(^{\sim}\)-closed sets contained in \(A\).

In symbols \(Ng^{\sim} - cl(A) = \cup \{B: B\text{ is Nano generalized}^{\sim}\text{-closed set and } B \subseteq A\}\).

**Proposition 4.5:** For any \(A \subseteq U\),

(i) \(Ng^{\sim}cl(A)\) is the smallest Ng\(^{\sim}\)-closed set containing \(A\).

(ii) \(A\) is Ng\(^{\sim}\)-closed if and only if Ng\(^{\sim}\)cl(A) = A

(iii) \(A \subseteq Ng^{\sim}cl(A) \subseteq cl(A)\)

**Proposition 4.6:** For any two subsets \(A\) and \(B\) of \(U\),

(i) If \(A \subseteq B\), then \(Ng^{\sim}cl(A) \subseteq Ng^{\sim}cl(B)\)

(ii) \(Ng^{\sim}cl(A \cup B) \subseteq Ng^{\sim}cl(A) \cap Ng^{\sim}cl(B)\)

**Theorem 4.7:**

A function \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano \(g^1\)-continuous if and only if the inverse image of every Nano closed set in \(V\) is Nano \(g^1\)-closed in \(U\).

**Proof:**

Let \(f\) be Nano \(g^1\)-continuous and \(F\) be Nano closed in \(V\).

That is \(V/F\) is Nano open in \(V\).

Since \(f\) is Nano \(g^1\)-continuous, \(f^{-1}(V - F)\) is Nano \(g^1\)-open in \(U\).

That is \(U - f^{-1}(V - F)\) is Nano \(g^1\)-closed in \(U\).

Therefore \(f^{-1}(F)\) is Nano \(g^1\)-closed in \(U\).

Thus the inverse image of every Nano closed set in \(V\) is Nano \(g^1\)-closed in \(U\).

If \(f\) is Nano \(g^1\)-continuous on \(U\).

Conversely, let the inverse image of every Nano closed set in \(V\) be Nano \(g^1\)-closed in \(U\).

Let \(G\) be Nano open in \(V\). Then \(V/G\) is Nano closed in \(V\).

Then \(f^{-1}(V - G)\) is Nano \(g^1\)-closed in \(U\).

That is \(U - f^{-1}(G)\) is Nano \(g^1\)-closed in \(U\). Therefore \(f^{-1}(G)\) is Nano \(g^1\)-open in \(U\).

By definition, \(f\) is Nano \(g^1\)-continuous.

**Theorem 4.8:**

Let \(U\) and \(V\) be any two Nano topological spaces. Let \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) be Nano continuous function. Then \(f\) is Nano \(g^1\)-continuous.

**Proof:**

Let \(B\) be any Nano open set in \(V\).

Since \(f\) is Nano continuous, \(f^{-1}(B)\) is Nano open in \(U\).

Since, every Nano open set is Ng\(^{\sim}\)-open set.

So \(f^{-1}(B)\) is Ng\(^{\sim}\)-open in \(U\).

Thus, the inverse image of every Nano open set is Nano \(g^1\)-open. Therefore \(f\) is Nano \(g^1\)-continuous.

**Remark 4.9:**

The converse of the above theorem need not be true as seen from the following example.

**Example 4.10:**

Let \(U = \{a, b, c, d\}\) with \(U/R = \{\{c\}, \{d\}, \{a, b\}\}\) and \(X = \{a, c\}\).
Then \( \tau_{g}(X) = \{ U, \Phi, \{ c \}, \{ a, b, c \}, \{ a,b \} \} \) which are open sets.
The Nano closed sets = \( \{ U, \Phi, \{ a,b,d \}, \{ d \}, \{ c,d \} \} \).
Nano \( g^\wedge \)-open sets are \( \{ \Phi, U, \{ a, b, c \}, \{ a, b \}, \{ b,c \}, \{ a,c \}, \{ c, a \}, \{ b \} \} \).

Let \( V = \{ a, b, c, d \} \) with \( U/R = \{ \{ a \}, \{ d \}, \{ b,c \} \} \) and \( Y = \{ a,b,c \} \).

Then, the Nano topology is defined as,
\[
\tau_{g}(X) = \{ U, \Phi, \{ a, b, c \}, \{ a,b \} \}
\]
then, \( f : (U, \tau_{g}(X)) \rightarrow (V, \tau_{g}(Y)) \) as \( f(c)= b, f(b)=d, f(a)=c, f(d)=a \).

Therefore, \( f \) is Nano semi-continuous.

Remark 4.15:
The converse of the above theorem need not be true as seen from the following example.

Example 4.16:
Let \( U = \{ a, b, c, d \} \) with \( U/R = \{ \{ c \}, \{ d \}, \{ a,b \} \} \) and \( X = \{ a, c \} \).

Then \( \tau_{g}(X) = \{ U, \Phi, \{ c \}, \{ a, b, c \}, \{ a,b \} \} \) which are open sets.
Nano \( g^\wedge \)-open sets are \( \{ \Phi, U, \{ a, b, c \}, \{ a, b \}, \{ b,c \}, \{ a,c \}, \{ c, a \}, \{ b \} \} \).

Let \( V = \{ a, b, c, d \} \) with \( U/R = \{ \{ a \}, \{ b, c,d \}, \{ a,b \} \} \) and \( Y = \{ a,b,c \} \).

Then, the Nano topology is defined as,
\[
\tau_{g}(X) = \{ U, \Phi, \{ a \}, \{ b, c,d \}, \{ a,b \} \}
\]
then, \( f : (U, \tau_{g}(X)) \rightarrow (V, \tau_{g}(Y)) \) as \( f(b)= a, f(c)=d, f(d)=b \).

Therefore \( f \) is Nano semi-continuous.

Theorem 4.11:
Every \( Ng \) continuous function is \( Ng^\wedge \) continuous function.

Proof:
Let \( f : (U, \tau_{g}(X)) \rightarrow (V, \tau_{g}(Y)) \) be \( Ng \) continuous and \( B \) be any Nano open set in \( (V, \tau_{g}(Y)) \).
Since \( f \) is \( Ng \) continuous, \( f^{-1}(B) \) is Nano \( g \)-open in \( U \).

Example 4.13:
Let \( U = \{ a, b, c, d \} \) with \( U/R = \{ \{ c \}, \{ d \}, \{ a,b \} \} \) and \( X = \{ a, c \} \).

Then \( \tau_{g}(X) = \{ U, \Phi, \{ c \}, \{ a, b, c \}, \{ a,b \} \} \) which are open sets.
The Nano closed sets = \( \{ U, \Phi, \{ a,b,d \}, \{ d \}, \{ c,d \} \} \).
Nano \( g^\wedge \)-open sets are \( \{ \Phi, U, \{ a, b, c \}, \{ a, b \}, \{ b,c \}, \{ a,c \}, \{ c, a \}, \{ b \} \} \).

Let \( V = \{ a, b, c, d \} \) with \( U/R = \{ \{ a \}, \{ d \}, \{ b,c \} \} \) and \( Y = \{ a,b,c \} \).

Then \( \tau_{g}(X) = \{ U, \Phi, \{ a, b, c \}, \{ a,b \} \} \) which are open sets.
Nano \( g^\wedge \)-open sets are \( \{ \Phi, U, \{ a, b, c \}, \{ a, b \}, \{ b,c \}, \{ a,c \}, \{ c, a \}, \{ b \} \} \).

Let \( V = \{ a, b, c, d \} \) with \( U/R = \{ \{ a \}, \{ b, c,d \}, \{ a,b \} \} \) and \( Y = \{ a,b,c \} \).

Then, the Nano topology is defined as,
\[
\tau_{g}(X) = \{ U, \Phi, \{ a \}, \{ b, c,d \}, \{ a,b \} \}
\]
then, \( f : (U, \tau_{g}(X)) \rightarrow (V, \tau_{g}(Y)) \) as \( f(b)= a, f(c)=d, f(d)=b, f(a)=d \).

Therefore, \( f \) is Nano semi-continuous.

Remark 4.12:
The converse of the above theorem need not be true as seen from the following example.

Example 4.19:
Let \( U = \{ a, b, c, d \} \) with \( U/R = \{ \{ c \}, \{ d \}, \{ a,b \} \} \) and \( X = \{ a, c \} \).

Then \( \tau_{g}(X) = \{ U, \Phi, \{ c \}, \{ a, b, c \}, \{ a,b \} \} \) which are open sets.
Nano \( g^\wedge \)-open sets are \( \{ \Phi, U, \{ a, b, c \}, \{ a, b \}, \{ b,c \}, \{ a,c \}, \{ c, a \}, \{ b \} \} \).

Let \( V = \{ a, b, c, d \} \) with \( U/R = \{ \{ a \}, \{ b, c,d \}, \{ a,b \} \} \) and \( Y = \{ a,b,c \} \).

Then, the Nano topology is defined as,
\[
\tau_{g}(X) = \{ U, \Phi, \{ a \}, \{ b, c,d \}, \{ a,b \} \}
\]
then, \( f : (U, \tau_{g}(X)) \rightarrow (V, \tau_{g}(Y)) \) as \( f(b)= a, f(c)=d, f(d)=b \).

Therefore, \( f \) is Nano semi-continuous.
Definition 4.20: Let x be a point of \((U, \tau_\Phi(X))\) and \(P\) be a subset of \(U\). Then \(P\) is called \(\Phi\)-neighborhood of \(x\) in \((U, \tau_\Phi(X))\), if there exist a \(\Phi\)-open set \(Q\) of \((U, \tau_\Phi(X))\) such that \(x \notin Q \subseteq P\).

Theorem 4.21: Let \(A\) be a subset of \((U, \tau_\Phi(X))\). Then \(x \in \Phi\text{-cl}(A)\) if and only if every \(\Phi\)-neighborhood \(W_x^\Phi\) of \(x\) in \((U, \tau_\Phi(X))\), \((A \cap W_x^\Phi) \neq \emptyset\).

Proof: Necessary part: Assume that \(x \in \Phi\text{-cl}(A)\). Then there exist \(\Phi\)-neighborhoods \(W_x^\Phi\) of \(x\) such that \((A \cap W_x^\Phi) \neq \emptyset\). Hence \(x \in \Phi\text{-cl}(A)\).

Sufficiency part: Assume that \(x \notin \Phi\text{-cl}(A)\). Then \((A \cap W_x^\Phi) = \emptyset\). Suppose \(x\) does not belong to \(\Phi\text{-cl}(A)\). Then \((A \cap W_x^\Phi) = \emptyset\), contradiction. Thus \(x \notin \Phi\text{-cl}(A)\).

Theorem 4.22: Let \(f: (U, \tau_\Phi(X)) \to (V, \tau_\Phi(Y))\) be a function. Then the following are equivalent.

(i) The function is \(\Phi\)-continuous.
(ii) The inverse of each \(\Phi\)-open set in \((V, \tau_\Phi(Y))\) is \(\Phi\)-open in \((U, \tau_\Phi(X))\).
(iii) For each \(x\) in \(U\), the inverse of every neighborhood of \(f(x)\) is a \(\Phi\)-neighborhood of \(x\).
(iv) For each \(x\) in \(U\) and each neighborhood \(W\) of \(f(x)\), there is a \(\Phi\)-neighborhood \(M\) of \(x\) such that \(f(M) \subseteq W\).
(v) For each subset \(A\) of \((U, \tau_\Phi(X))\), \(f(\Phi\text{-cl}(A)) \subseteq \Phi\text{-cl}(f(A))\).
(vi) For each subset \(B\) of \((V, \tau_\Phi(Y))\), \(\Phi\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\Phi\text{-cl}(B))\).

Proof: (i) \(\Leftrightarrow\) (ii): Let \(f: (U, \tau_\Phi(X)) \to (V, \tau_\Phi(Y))\) be \(\Phi\)-continuous and \(B\) be an \(\Phi\)-open set in \((V, \tau_\Phi(Y))\). Then \(B^\Phi\) is \(\Phi\)-closed in \(V\). Hence by the definition of \(\Phi\)-continuous function, \(f^{-1}(B^\Phi) = f^{-1}(B)\) is \(\Phi\)-closed in \(U\). Thus \(f^{-1}(B)\) is \(\Phi\)-open in \(U\).

Conversely, assume that \(f^{-1}(B)\) is \(\Phi\)-open in \(U\) for each \(\Phi\)-open set \(B\) in \(V\). Let \(F\) be a \(\Phi\)-closed set in \((V, \tau_\Phi(Y))\). Then \(F^\Phi\) is \(\Phi\)-open in \(V\) and by assumption \(f^{-1}(F^\Phi) = f^{-1}(F)\) is \(\Phi\)-closed in \(U\). So \(f\) is \(\Phi\)-continuous.

(ii) \(\Rightarrow\) (iii): For \(x \in (U, \tau_\Phi(X))\), let \(W\) be the neighborhood of \(f(x)\). Then there exist an \(\Phi\)-open set \(B\) in \((V, \tau_\Phi(Y))\) such that \(f(x) \in B \subseteq W\).

Consequently, \(f^{-1}(B)\) is \(\Phi\)-open in \((U, \tau_\Phi(X))\) and \(x \in f^{-1}(f^{-1}(B)) \subseteq f^{-1}(W)\). Thus \(f^{-1}(W)\) is \(\Phi\)-neighborhood in \((U, \tau_\Phi(X))\).

(iii) \(\Rightarrow\) (iv): Let \(x \in U\) and \(W\) be the neighborhood of \(f(x)\). Then by assumption, \(M = f^{-1}(W)\) is a \(\Phi\)-neighborhood of \(x\) and \(f(M) = f^{-1}(f^{-1}(W)) \subseteq W\).

(iv) \(\Leftrightarrow\) (v): Suppose (iv) holds and let \(y \in f(\Phi\text{-cl}(A))\) and let \(W\) be the neighborhood of \(y\). Then there exist \(x \in U\) and a \(\Phi\)-neighborhood \(M\) of \(x\) such that \(f(M) = f^{-1}(W) \subseteq W\).

Since \(x \in \Phi\text{-cl}(A)\), by previous theorem, \((M \cap A) \neq \emptyset\) and hence \(f(M) \cap W) \neq \emptyset\).

Hence \(y = f(x) \in \Phi\text{-cl}(f(A))\). That is \(f(\Phi\text{-cl}(A)) \subseteq \Phi\text{-cl}(f(A))\).

Conversely, suppose that (v) holds and let \(x \in U\) and \(W\) be the neighborhood of \(f(x)\).

Let \(A = f^{-1}(W)\). Since \(f(\Phi\text{-cl}(A)) \subseteq \Phi\text{-cl}(f(A))\), \(W\) we have \(\Phi\text{-cl}(A) = A\).

Since \(x\) does not belong to \(\Phi\text{-cl}(A)\), there exist a \(\Phi\)-neighborhood \(M\) of \(x\) such that \((M \cap A) = \emptyset\) and \(f(M) \subseteq f(U) \subseteq W\).

(v) \(\Leftrightarrow\) (vi): Suppose (v) holds and \(B\) be any subset of \((V, \tau_\Phi(Y))\). Then replacing \(A\) by \(f^{-1}(B)\) in (v), we have \(f(\Phi\text{-cl}(f^{-1}(B))) \subseteq \Phi\text{-cl}(f^{-1}(B)) \subseteq \Phi\text{-cl}(B)\).

That is \(\Phi\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\Phi\text{-cl}(B))\). Conversely, suppose that (vi) holds and let \(B = f(A)\) where \(A\) is the subset of \((U, \tau_\Phi(X))\). Then we have \(\Phi\text{-cl}(A) \subseteq \Phi\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\Phi\text{-cl}(B)) \subseteq f^{-1}(\Phi\text{-cl}(f(A)))\) which implies \(f(\Phi\text{-cl}(A)) \subseteq \Phi\text{-cl}(f(A))\).

This completes the proof of the theorem.

Theorem 4.23: For each \(x \in (U, \tau_\Phi(X))\) and for each neighborhood \(B\) of \(f(x)\), if there is a \(\Phi\)-neighborhood \(M\) of \(x\) such that \(f(M) \subseteq B\), then for each point \(x\) in \((U, \tau_\Phi(X))\) and each \(\Phi\)-open set \(B\) in \((V, \tau_\Phi(Y))\) with \(f(x) \in B\), there is a \(\Phi\)-open set \(A\) in \(U\) such that \(x \in A\), \(f(A) \subseteq B\).

Proof: For \(x \in (U, \tau_\Phi(X))\), let \(B\) be a \(\Phi\)-open set containing \(f(x)\). Then \(B\) is a neighborhood of \(f(x)\).

So by assumption, there exists \(\Phi\)-neighborhood \(M\) of \(x\), such that \(f(M) \subseteq B\).

Hence there exists a \(\Phi\)-open set \(A\) in \(U\) such that \(x \in A \subseteq M\).

So \(f(A) \subseteq f(M) \subseteq B\).

That is \(f(A) \subseteq B\).

Theorem 4.24: For each subset \(A\) of \((U, \tau_\Phi(X))\), \(f(\Phi\text{-cl}(A)) \subseteq \Phi\text{-cl}(f(A))\) if and only if the inverse of each \(\Phi\)-closed set in \(V\) is \(\Phi\)-closed set in \(U\).

Proof: Suppose the inverse of each \(\Phi\)-closed set is \(\Phi\)-closed set. Let \(A\) be a subset of \(U\).

Since \(A \subseteq f^{-1}(f(A))\), we have \(A \subseteq f^{-1}(\Phi\text{-cl}(f(A)))\).

Since \(\Phi\text{-cl}(f(A))\) is \(\Phi\)-closed in \(V\), by assumption \(f^{-1}(\Phi\text{-cl}(f(A)))\) is \(\Phi\)-closed set containing \(A\).

Also \(\Phi\text{-cl}(A) \subseteq f^{-1}(\Phi\text{-cl}(f(A)))\).

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Thus $f(\text{Ncl}(A)) \subseteq \text{f}^{-1}(\text{Ncl}(f(A))) \subseteq \text{Ncl}(f(A))$.

Conversely, suppose the inverse of every Nano closed set in Nano $g^\circ$-closed.

Let $B$ be a closed subset of $(V, \tau_g(Y))$.

Then by assumption, $f(\text{Ng}^\circ\text{cl}(f^{-1}(B))) \subseteq \text{Ncl}(\text{f}(f^{-1}(B))) \subseteq N\text{cI}(B) \subseteq B$.

That is $\text{Ng}^\circ\text{cl}(f^{-1}(B)) \subseteq f^{-1}(B)$ which implies $f^{-1}(B)$ is Ng closed.

**Definition 4.25:**
Let $(U, \tau^g(U))$ and $(V, \tau^g(V))$ be two nano topological spaces. Then a function $f: U \rightarrow V$ is said to be nano generalized $^g$-irresolute (nano $g^\circ$-irresolute) if the inverse image of every nano $g^\circ$ open set in $V$ is nano $g^\circ$-open in $U$.

**Example 4.26:**
Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b\}, \{d\}\}$ and $V = \{a, d\}$.

Then the nano topology is defined as $\tau^g(U) = \{\{a\}, \{b\}, \{d\}, \{a, c\}, \{a, d\}\}$.

Let $V = \{x, y, z, w\}$ with $V/R = \{\{x\}, \{y, z\}, \{w\}\}$.

Then $\tau^g(V) = \{V, \phi, \{x\}, \{x, y, z\}, \{y, z\}\}$.

Define $f:U \rightarrow V$ as $f(a) = x, f(b) = w, f(c) = y$ and $f(d) = z$.

Then $f$ is nano $g^\circ$-irresolute since the inverse image of every nano $g^\circ$ open set in $V$ is nano $g^\circ$-open in $U$.

**Theorem 4.27:**
A function $f: U \rightarrow V$ is nano $g^\circ$-irresolute, then $f$ is nano $g^\circ$-continuous.

**Proof:**
Since every nano open set is Ng $g^\circ$ open.

The inverse image of every nano open set in $V$ is nano $g^\circ$-open in $U$; whenever the inverse image of every nano $g^\circ$-open set is nano $g^\circ$-open.

Therefore, any nano $g^\circ$-irresolute function is nano $g^\circ$-continuous.

**Remark 4.28:**
The converse of the above theorem need not be true shown in the following example.

**Example 4.29:**
Let $U = \{a, b, c, d\}$ with $R/U = \{\{b\}, \{c\}, \{a, d\}\}$ and $X = \{a, c\}$.

Then the nano topology is defined as $\tau^g(X) = \{U, \phi, \{c\}, \{a, c\}, \{a, d\}\}$.

Let $V = \{x, y, z, w\}$ with $V/R = \{\{x\}, \{y, z\}, \{w\}\}$.

Then $\tau^g(V) = \{V, \phi, \{x\}, \{x, y, z\}, \{y, z\}\}$.

Define $f:U \rightarrow V$ as $f(a) = x, f(b) = w, f(c) = y$ and $f(d) = z$.

Then $f$ is nano $g^\circ$-irresolute.

But $f$ is not nano $g^\circ$-continuous since $f^{-1}(\{x\}) = \{c\}$ which is not nano $g^\circ$-open in $V$ whereas $\{z\}$ is nano $g^\circ$-open in $V$.

Thus a nano $g^\circ$-continuous function is not nano $g^\circ$-irresolute.

**Theorem 4.30:**
If $f:U \rightarrow V$ is Ng $g^\circ$-continuous and $g:V \rightarrow W$ is Nano continuous.

Then $g \circ f: U \rightarrow W$ is Nano $g^\circ$-continuous.

**Proof:**
Let $G$ be nano open in $W$.

Then $g^{-1}(G)$ is nano open in $V$.

Since $g$ is Nano continuous.

Thus $g^{-1}(G)$ is nano $g^\circ$-open in $V$.

Since every nano open is nano $g^\circ$-open.

Then $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is nano $g^\circ$-open in $U$ and hence $g \circ f$ is nano $g^\circ$-continuous.

**Theorem 4.31:**
If $f:U \rightarrow V$ is Ng $g^\circ$-irresolute and $g:V \rightarrow W$ is Nano $g^\circ$-continuous.

Then $g \circ f: U \rightarrow W$ is Nano $g^\circ$-continuous.

**Proof:**
Let $G$ be nano open in $W$.

Then $g^{-1}(G)$ is nano $g^\circ$ open in $V$.

Since $g$ is nano $g^\circ$ continuous.

Thus $g^{-1}(G)$ is nano $g^\circ$-open in $V$.

Since every nano open is nano $g^\circ$-open.

Then $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is nano $g^\circ$-open in $U$ and hence $g \circ f$ is nano $g^\circ$-continuous.

**Theorem 4.32:**
If $f:U \rightarrow V$ is Ng $g^\circ$-irresolute and $g:V \rightarrow W$ is Nano $g^\circ$-continuous.

Then $g \circ f: U \rightarrow W$ is Nano $g^\circ$-continuous.

**Proof:**
Let $G$ be nano open in $W$.

Then $g^{-1}(G)$ is nano $g^\circ$-open in $V$.

Since $g$ is nano $g^\circ$ continuous.

Thus $g^{-1}(G)$ is nano $g^\circ$-open in $V$.

Since every nano open is nano $g^\circ$-open.

Then $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is nano $g^\circ$-open in $U$ and hence $g \circ f$ is nano $g^\circ$-continuous.

**Theorem 4.33:**
Let $(U, \tau^g(U)), (U, \tau^g(Z))$ be Nano topological spaces. If $f:U \rightarrow V$ and $g:V \rightarrow W$ are Nano $g^\circ$-irresolute.

Then $g \circ f: U \rightarrow W$ is Nano $g^\circ$ continuous.

**Proof:**
Let $G \subseteq W$ be Nano $g^\circ$ open.

Then $g^{-1}(G)$ is Nano $g^\circ$ open in $V$.

Since, $g$ is Nano $g^\circ$-irresolute.

Then $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is nano $g^\circ$-open in $U$.

Therefore $g \circ f$ is Ng $g^\circ$-irresolute.

**4. REFERENCES:**


