Collocation Method for Ninth Order Boundary Value Problems by Quintic B-splines

Abstract:
A finite element method involving collocation method with quintic B-splines as basic functions has been developed to solve ninth order boundary value problems. The ninth, eight, seventh, sixth and fifth order derivatives for the dependent variable is approximated by the finite differences. The basic functions are redefined into a new set of basic functions which in number match with the number of collocated points selected in the space variable domain. The proposed method is tested one linear and one nonlinear boundary value problems. The solution to a nonlinear problem has been obtained by the limit of a sequence of solutions of linear problems generated by the quasilinearization technique. Numerical results obtained by the present method are in good agreement with the exact solutions available in the literature.

Keywords: Absolute error, Basis function, Collocation method, Ninth order boundary value problem, Quintic B-spline.

I. INTRODUCTION
In this paper, we consider a general ninth order boundary value problem

\[ a_9(x)y^{(9)}(x) + a_8(x)y^{(8)}(x) + a_7(x)y^{(7)}(x) + a_6(x)y^{(6)}(x) + a_5(x)y^{(5)}(x) + a_4(x)y^{(4)}(x) + a_3(x)y''(x) + a_2(x)y'(x) + a_1(x)y(x) = b(x), c < x < d \]

(1) subject to boundary conditions

\[ y(c) = A_0, \quad y(d) = C_0, \quad y'(c) = A_1, \quad y'(d) = C_1, \quad y''(c) = A_2, \quad y''(d) = C_2, \quad y^{(4)}(c) = A_4, \quad y^{(4)}(d) = C_4 \]

(2) where \( a_0, a_1, a_2, a_3, a_4, C_0, C_1, C_2, C_4 \) are finite real constants and \( a_0(x), a_1(x), a_2(x), a_3(x), a_4(x), a_5(x), a_6(x), a_7(x), a_8(x), a_9(x) \) and \( b(x) \) are all continuous functions defined on the interval \([c, d]\). The ninth-order boundary value problems are known to arise in the study of astrophysics, hydrodynamic and hydro magnetic stability [5]. A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydromagnetic stability [1]. The existence and uniqueness of the solution for these types of problems have been discussed by Agarwal [2]. Finding the analytical solutions of such type of boundary value problem in general is not possible. Over the years, many researchers have worked on ninth-order boundary value problems by using different methods for numerical solutions. Chawla and Katti [3] developed a finite difference scheme for the solution of a special case of nonlinear higher order two point boundary value problems. Wazwaz [4] developed the solution of a special type of higher order boundary value problems by using the modified Adomian decomposition method. Hassan and Erturk [5] provided solution of different types of linear and nonlinear higher order boundary value problems by using Differential transformation method. Tauseef and Ahmet [6] presented the solution of ninth and tenth order boundary value problems by using homotopy perturbation method without any discretization, linearization or restrictive assumptions. Tauseef and Ahmet [7] developed modified variational method for solving ninth and tenth order boundary value problems introducing He’s polynomials in the correction functional. Jafar and Shirin [8] presented homotopy perturbation method for solving the boundary value problems of higher order by reformulating them as an equivalent system of integral equations. Tawfiq and Yassen [9] developed semi-analytic technique for the solution of higher order boundary value problems using two-point oscillatory interpolation to construct polynomial solution. Hosain and Islam [10] presented the Galerkin method with Legendre polynomials as basic functions for the solution of odd higher order boundary value problems. Samir [11] developed spectral collocation method for the solution of \( m \)th order boundary value problems with help of Chebyshev polynomials by converting the given differential equation into a system of first order boundary value problems. So far, ninth order boundary value problems have not been solved by using collocation method with quintic B-splines as basis functions. In this paper, we try to present a simple finite element method which involves collocation approach with quintic B-splines as basis functions to solve the ninth order boundary value problem of the type (1)-(2).

This paper is organized as follows. In section II of this paper, the justification for using the collocation method has been mentioned. In section III, the definition of quintic B-splines has been described. In section IV, description of the collocation method with quintic B-splines as basis functions has been presented and in section V, solution procedure to find the nodal parameters is presented. In section VI, numerical examples of both linear and non-linear boundary value problems are presented. The solution to a nonlinear problem has been obtained as the limit of a sequence of solutions of linear problems generated by the quasilinearization technique [12]. Finally, the last section is dealt with conclusions of the paper.

II. JUSTIFICATION USING COLLOCATION METHOD
In finite element method (FEM) the approximate solution can be written as a linear combination of basic functions which
constitute a basis for the approximation space under consideration. FEM involves variational methods like Ritz approach, Galerkins approach, least squares method and collocation method etc. The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points in the given space variable domain where N is the number of basis functions in the basis [13]. That means, to get an accurate solution by the collocation method one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain.

Further, the collocation method is the easiest to implement among the variational methods of FEM. When a differential equation is approximated by mth order B-splines, it yields (m+1)th order accurate results [13]. Hence this motivated us to solve a ninth order boundary value problem of type (1)-(2) by collocation method with quintic B-splines as basis functions.

### III. Definition of Quintic B-Splines

The quintic B-splines are defined in [14–16]. The existence of quintic spline interpolate \(s(x)\) to a function in a closed interval, \([a, b]\) for spaced knots (need not be evenly spaced) of a partition \(x_0 < x_1 < \ldots < x_{n+1} = d\) is established by constructing it. The construction of \(s(x)\) is done with the help of the quintic B-splines. Introduce ten additional knots \(x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{1}, x_{2}, x_{3}, x_{4}, x_{5}\) and \(x_{n+5}\) in such a way that

- \(x_{-5} < x_1 < x_2 < x_3 < x_4 < x_5\)
- \(x_{n+5} < x_n < x_{n+2} < x_{n+3} < x_{n+4}\)

Now the quintic B-splines \(B_j(x)\)s are defined by

\[
B_j(x) = \begin{cases} 
\sum_{i=j}^{j+5} \left( x - x_i \right)^5 \pi(x), & x \in [x_{j-2}, x_{j+2}] \\
0, & \text{otherwise}
\end{cases}
\]

where

\[
\pi(x) = \prod_{i=j}^{j+5} (x - x_i)
\]

and

\[
(x_i - x) = \begin{cases} 
(x_i - x)^5, & \text{if } x_i \geq x \\
0, & \text{if } x_i \leq x
\end{cases}
\]

where \(B_2(x), B_3(x), \ldots, B_{n+4}(x), B_{n+5}(x)\) forms a basis for the space \(S_n(x)\) of quintic polynomial splines. Schoenberg [16] has proved that quintic B-splines are the unique nonzero splines of smallest compact support with the knots at \(x_0 < x_1 < x_2 < x_3 < x_4 < x_5\) and \(x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4}\). Description of the method

To solve the boundary value problem (1) subject to boundary conditions (2) by the collocation method with quintic B-splines as basic functions, we define the approximation for \(y(x)\) as

\[
y(x) = \sum_{j=1}^{n+2} \alpha_j B_j(x)
\]

where \(\alpha_j\)'s are the nodal parameters to be determined and \(B_j(x)\)'s are sextic B-spline basis functions. In the present method, the mesh points \(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}\) are selected as the collocation points. In collocation method, the number of basic functions in the approximation should match with the number of collocation points [13]. Here the number of basic functions in the approximation (3) is \(n+5\), where as the number of selected collocation points is \(n+4\). So, there is a need to redefine the basic functions into a new set of basic functions which in number match with the number of selected collocation points. The procedure for redefining the basic functions are as follows:

Using the definition of quintic B-splines, the Dirichlet, Neumann, second order derivatives boundary conditions, third order derivative boundary conditions and fourth order derivative boundary condition of (2), we get the approximate solution at the boundary points as

\[
y(c) = y(x_0) = \sum_{j=1}^{n+2} \alpha_j B_j(x_0) = A_0
\]

\[
y(d) = y(x_1) = \sum_{j=1}^{n+2} \alpha_j B_j(x_1) = C_0
\]

\[
y'(c) = y'(x_0) = \sum_{j=1}^{n+2} \alpha_j B'_j(x_0) = A_1
\]

\[
y'(d) = y'(x_1) = \sum_{j=1}^{n+2} \alpha_j B'_j(x_1) = C_1
\]

\[
y''(c) = y''(x_0) = \sum_{j=1}^{n+2} \alpha_j B''_j(x_0) = A_2
\]

\[
y''(d) = y''(x_1) = \sum_{j=1}^{n+2} \alpha_j B''_j(x_1) = C_2
\]

\[
y'''(c) = y'''(x_0) = \sum_{j=1}^{n+2} \alpha_j B'''_j(x_0) = A_3
\]

\[
y'''(d) = y'''(x_1) = \sum_{j=1}^{n+2} \alpha_j B'''_j(x_1) = C_3
\]

\[
y^{(4)}(c) = y^{(4)}(x_0) = \sum_{j=1}^{n+2} \alpha_j B^{(4)}_j(x_0) = A_4
\]

Eliminating \(\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\) and \(\alpha_{n+1}\) from the equations (3) to (12), we get the approximation for \(y(x)\) as

\[
y(x) = w(x) + \sum_{j=3}^{n+2} \alpha_j T_j(x)
\]

where

\[
w(x) = w_1(x) + \frac{A_2 - w^{(4)}(x_0)}{S_2''(x_0)} S_2(x)
\]

\[
w_1(x) = w_1(x) + \frac{A_1 - w_1''(x_0)}{R_1''(x_0)} R_1(x) + \frac{C_1 - w_1''(x_0)}{Q_1''(x_0)} Q_1(x)
\]

\[
w_2(x) = w_2(x) + \frac{A_2 - w_2''(x_0)}{Q_2''(x_0)} Q_0(x) + \frac{C_2 - w_2''(x_0)}{Q''(x_0)} Q(x)
\]

\[
w_3(x) = w_3(x) + \frac{A_3 - w_3''(x_0)}{P_3''(x_0)} P_2(x) + \frac{C_3 - w_3''(x_0)}{P''(x_0)} P_1(x)
\]

\[
w_4(x) = \frac{A_4}{B_2(x)} B_2(x) + \frac{C_0}{B_{n+2}(x)} B_{n+2}(x)
\]

\[
T_j(x) = S_j(x), \quad j = 3, 4, \ldots, n-2
\]
Now applying the collocation method to (1), we get
\[
\begin{align*}
\sum_{j=3}^{n-1} a_j (y_{i,j}) & = b(x_i) \quad \text{for } i = 3, 4, \ldots, n-2 \quad \text{(20)} \\
\end{align*}
\]
where
\[
y_i = y(x_i) = w(x_i) + \sum_{j=3}^{n-1} a_j T_j(x_i)
\]

Now applying the collocation method to (1), we get
\[
\begin{align*}
\sum_{j=3}^{n-1} a_j (y_{i,j}) & = b(x_i) \quad \text{for } i = 3, 4, \ldots, n-2.
\end{align*}
\]

Rearranging the terms and writing the system of equations (23) and (24) in matrix form, we get
\[
A \alpha = B\quad \text{(25)}
\]
where \(A = [a_{ij}]\).
\[ a_y = \frac{a_y(x)}{2h} \left[ T_y^{(4)}(x_{i+1}) + 4T_y^{(4)}(x_i) - 6T_y^{(4)}(x_{i-1}) + 4T_y^{(4)}(x_{i-2}) - T_y^{(4)}(x_{i-3}) \right] \]

\[ b_y = \frac{a_y(x)}{2h} \left[ w^{(4)}(x_{i+1}) - 5w^{(4)}(x_i) + 10w^{(4)}(x_{i-1}) - 10w^{(4)}(x_{i-2}) - w^{(4)}(x_{i-3}) \right] \]

for \( i = 3, 4, \ldots, n-1 \); \( j = 3, 4, \ldots, n-2 \). (26)

\[ b_y = \frac{a_y(x)}{h^2} \left[ w_y^{(4)}(x_{i+1}) + 4w_y^{(4)}(x_i) - 6w_y^{(4)}(x_{i-1}) + 4w_y^{(4)}(x_{i-2}) - w_y^{(4)}(x_{i-3}) \right] \]

\[ a_y = \frac{a_y(x)}{h^2} \left[ w_y^{(4)}(x_{i+1}) + 4w_y^{(4)}(x_i) - 6w_y^{(4)}(x_{i-1}) + 4w_y^{(4)}(x_{i-2}) - w_y^{(4)}(x_{i-3}) \right] \]

for \( i = n-2; j = 3, 4, \ldots, n-2 \). (27)

\[ b_y = \frac{a_y(x)}{2h} \left[ w^{(4)}(x_{i+1}) - 5w^{(4)}(x_i) + 10w^{(4)}(x_{i-1}) - 10w^{(4)}(x_{i-2}) - w^{(4)}(x_{i-3}) \right] \]

\[ a_y = \frac{a_y(x)}{h^2} \left[ w_y^{(4)}(x_{i+1}) + 4w_y^{(4)}(x_i) - 6w_y^{(4)}(x_{i-1}) + 4w_y^{(4)}(x_{i-2}) - w_y^{(4)}(x_{i-3}) \right] \]

for \( i = 3, 4, \ldots, n-1 \). (28)

\[ \alpha = \left[ \alpha_1, \alpha_2, \ldots, \alpha_n \right]^T. \]

\section*{IV. Solution Procedure to Find the Nodal Parameters}

The basis function \( T_i(x) \) is defined only in the interval \([x_{i-2}, x_{i+2}]\) and outside of this interval it is zero. Also at the end points of the interval \([x_{i-2}, x_{i+2}]\) the basis function \( T_i(x) \) vanishes. Therefore, \( T_i(x) \) is having non-vanishing values at the mesh points \( x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+n-2} \) and zero at the other mesh points. The first four derivatives of \( T_i(x) \) also have the same nature at the mesh points as in the case of \( T_i(x) \). Using these facts, we can say that the Thus the stiffness matrix \( A \) is a thirteen diagonal band matrix. Therefore, the system of equations (25) is a thirteen band system in \( \alpha \)'s.

The nodal parameters \( \alpha \)'s can be obtained by using band matrix solution package. We have used the FORTRAN-90 programming to solve the boundary value problem (1)-(2) by the proposed method.

\section*{V. Numerical Results}

To demonstrate the applicability of the proposed method for solving the ninth order boundary value problems of the type (1) and (2), we considered one linear and one nonlinear boundary value problems. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

\textbf{Example 2:} Consider the linear boundary value problem

\[ y^{(9)} + y^{(7)} + xy^{(4)} + y''' + \sin x \ y' + y = 5x \sin x - \cos x + x^2 \cos x - x \sin^2 x \]

\[ + \sin x \cos x + x \cos x, \quad 0 < x < 1 \]

subject to

\[ y(0) = 0, y(1) = \cos 1, y'(0) = 1, y'(1) = \cos 1 - \sin 1, \]

\[ y''(0) = 0, y''(1) = -2 \sin 1 - \cos 1, \]

\[ y'''(0) = -3, y'''(1) = -3 \cos 1 + \sin 1, y^{(4)} (0) = 0. \]
The exact solution for the above problem is $y = x \cos x$.

The proposed method is tested on this problem where the domain $[0, 1]$ is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table I. The maximum absolute error obtained by the proposed method is $1.086163 \times 10^{-5}$. 

Table I

<table>
<thead>
<tr>
<th>x</th>
<th>Absolute error by the proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.947186E-06</td>
</tr>
<tr>
<td>0.2</td>
<td>2.312792E-06</td>
</tr>
<tr>
<td>0.3</td>
<td>2.589558E-06</td>
</tr>
<tr>
<td>0.4</td>
<td>1.847744E-06</td>
</tr>
<tr>
<td>0.5</td>
<td>1.159406E-06</td>
</tr>
<tr>
<td>0.6</td>
<td>2.457069E-06</td>
</tr>
<tr>
<td>0.7</td>
<td>1.445607E-05</td>
</tr>
<tr>
<td>0.8</td>
<td>2.014637E-06</td>
</tr>
<tr>
<td>0.9</td>
<td>1.835823E-06</td>
</tr>
</tbody>
</table>

Example 2: Consider the nonlinear boundary value problem

$$y^{(9)} - y'^{2} = \cos^{3} x, \quad 0 < x < 1 \quad (31)$$

subject to

$$y(0) = 0, y'(0) = 1, y'(1) = \cos 1, \quad y''(0) = 0, \quad y''(1) = -\sin 1, \quad y''''(0) = -1, \quad y''''(1) = -\cos 1, \quad y'''''(0) = 0.$$

The exact solution for the above problem is $y = \sin x$.

The nonlinear boundary value problem (31) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [12] as

$$y^{(9)}_{(n+1)} - y'^{2}_{(n+1)} = \cos^{3} x - 2y'^{2}_{(n)}y'_{(n)} \quad n = 0, 1, 2, ... \quad (32)$$

subject to

$$y_{(n+1)}(0) = 0, \quad y'_{(n+1)}(0) = \sin l, \quad y'_{(n+1)}(1) = \cos l,$$

$$y''_{(n+1)}(0) = 0, \quad y''_{(n+1)}(1) = -\sin l,$$

$$y'''_{(n+1)}(0) = -1, \quad y'''_{(n+1)}(1) = -\cos l, \quad y''''_{(n+1)}(0) = 0.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain $[0, 1]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (32). The obtained numerical results for this problem are presented in Table II. The maximum absolute error obtained by the proposed method is $1.445607 \times 10^{-5}$.

Table II

<table>
<thead>
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VI. CONCLUSIONS

In this paper, we have developed a collocation method with quintic B-splines as basis functions to solve ninth order boundary value problems. Here we have taken mesh points $x_{1}, x_{2}, ..., x_{n-1}, x_{n}$ as the collocation points. The septic B-spline basis set has been redefined into a new set of basis functions which in number match with the number of selected collocation points. The proposed method is applied to solve linear and non-linear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple method to solve a ninth order boundary value problem and its easiness for implementation.

VII. REFERENCES


