On the Quintic Non-Homogeneous Diophantine Equation

\[ x^4 - y^4 = 40(z^2 - w^2)p^3 \]

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Abstract:
The non-homogeneous quintic diophantine equation having five unknowns given by \( x^4 - y^4 = 40(z^2 - w^2)p^3 \) is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions and the special numbers are also presented.

Keywords: integral solutions, non-homogeneous, quintic diophantine equation.

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I. INTRODUCTION

While individual equations present a kind of puzzle and have been considered throughout history, the formulation of general theories of diophantine equations was an achievement of the twentieth century. Detailed and clear study on diophantine equations can be seen from [1-3]. For various problems and ideas on the quintic diophantine equations one may refer [4-9]. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous quintic diophantine equation with five unknowns given by

\[ x^4 - y^4 = 40(z^2 - w^2)p^3. \]

Also, a few interesting relations among the solutions and some special numbers like Jacobsthal, Mersenne, Kynea, Resel, Thabit-ibn-kurrah, Woodall, Carol, Sierpinski, Polygonal and Pyramidal numbers are presented.

II. NOTATIONS

- \( T_{m,n} \) = Triangular Number of rank \( n \).
- \( P_n^m \) = Pyramidal Number of rank \( n \).
- \( P(n) \) = Pronic Number of rank \( n \).
- \( Star_n \) = Star Number of rank \( n \).
- \( CH_n \) = Centered Hexagonal Number of rank \( n \).
- \( CC_n \) = Centered Cube Number of rank \( n \).
- \( RD_n \) = Rhombic Dodecagonal Number of rank \( n \).
- \( Gno_n \) = Gnomonic Number of rank \( n \).
- \( J_n \) = Jacobsthal Number of rank \( n \).
- \( H_n \) = Hex Number of rank \( n \).
- \( W_n \) = Woodall Number of rank \( n \).
- \( HRD_n \) = Hauy Rhombic Dodecagonal Number of rank \( n \).
- \( SO_n \) = Stella octangula Number of rank \( n \).
- \( TT_n \) = Truncated tetrahedron Number of rank \( n \).

III. METHOD OF ANALYSIS

The considered quintic diophantine equation with five unknowns is

\[ x^4 - y^4 = 40(z^2 - w^2)p^3 \] (1)

Introducing the linear transformations,

\[
\begin{align*}
x &= u + v \\
y &= u - v \\
z &= 2u + v \\
w &= 2u - v \\
\end{align*}
\] (2)

in (1) and simplifying we get

\[ u^2 + v^2 = 40p^3 \] (3)

Solving the above equation in different methods, we obtain different patterns of infinite number of integer solutions of equation (1)

PATTERN-I:

Let \( p = a^2 + b^2 \) (4)

Write 40 = (6 + 2i)(6 - 2i) (5)

Using (4) and (5) in (3) and using the method of factorization, Defint

\[(u + iv)(u - iv) = (6 + 2i)(6 - 2i)[(a + ib)(a - ib)]^3 \]

Equating real and imaginary parts we get

\[
\begin{align*}
u &= 6a^3 + 2b^3 - 18ab^2 - 6a^2b \\
v &= 2a^3 - 6b^3 - 6ab^2 + 18a^2b \\
\end{align*}
\]

Employing the values of \( u \) and \( v \) in (2) we get the non-zero distinct integer solutions of (1) to be

\[
\begin{align*}
x &= 8a^3 - 4b^3 - 24ab^2 + 12a^2b \\
y &= 4a^3 + 8b^3 - 12ab^2 - 24a^2b \\
z &= 14a^3 - 2b^3 - 42ab^2 + 6a^2b \\
w &= 10a^3 + 10b^3 - 30ab^2 - 30a^2b \\
p &= a^2 + b^2 \\
\end{align*}
\]
OBSERVATIONS:

1. \( \frac{y(a,a) \cdot z(a,a)}{2a^6} - 1 \) represents Kynea number

2. \( \frac{x(a,a) \cdot w(a,a) - 97a^6}{a^6} \) represents Carol number

3. \( 3.6(y(a,a) \cdot z(a,a)) \) represents Nasty number

4. \( -\frac{[y(a,a)]}{a^3} + 1 \) represents Thabit - ibn - karrabah number

5. \( \frac{x(a,a) \cdot w(a,a)}{a^6} + 63 \) represents Woodall number

6. \( p(2^n,1) - 2 = 3J_{2n} \)

PATTERN 2:

Write (3) as

\[ u^2 + v^2 = 40p^3, 1 \] (6)

and take

\[ 1 = \frac{(6+8i)(6-8i)}{10^2} \] (7)

Using (4), (5) and (7) in (6) and proceeding as in pattern 1 we get the non-zero distinct integer solutions of (1) to be

\[ x = 8a^3 + 4b^3 - 12a^2b - 24ab^2 \]
\[ y = -4a^3 + 8b^3 - 24a^2b + 12ab^2 \]
\[ z = 10a^3 + 10b^3 - 30a^2b - 30ab^2 \]
\[ w = -2a^3 + 14b^3 - 42a^2b + 6ab^2 \]
\[ p = a^2 + b^2 \]

OBSERVATIONS:

1. \( x(a,L) + y(a,L) + 10H_a - [RD_a + 2Pr(a) + 6Gno_a] = 0 \) (mod 27)

2. \( \text{HRD}_a + \text{Star}_a - [26Gno_a + 3SO_a + z(a,L)] = 0 \) (mod 13)

3. \( 3.6P_a^6 + H_b - 17Gno_a - w(1,b) + 3 = W_a \)

4. \( 4w(a,L) + 2p(a,L) + CC_a + CH_a + 6FT_a + 2T_{19,a} + 18Gno_a - 1 = \text{Kynea no.} \)

5. \( x(a,L) + w(a,L) + 45Pr(a) - [10Gno_a + RD_a + CC_a + 7] = W_a \)

The solutions presented in pattern 1 and pattern 2 are non-trivial only when \( a \neq b \)

PATTERN 3:

Write \( 1 = \frac{(4 + 3i)(4 - 3i)}{5^2} \) (8)

Using (4), (5) and (8) in (6) and proceeding as in pattern 2 we get

\[ u = \frac{1}{5} [18a^3 + 26b^3 - 54ab^2 - 78a^2b] \]
\[ v = \frac{1}{5} [26a^3 - 18b^3 - 78ab^2 + 54a^2b] \]

Since our interest is to find integer solutions, taking \( a = 5A; b = 5B \), in the above equations and proceeding as in earlier patterns we get the distinct non-zero integer solutions of (1) to be

\[ x = 1100A^3 + 200B^3 - 3300AB^2 - 600A^2B \]
\[ y = -200A^3 + 1100B^3 + 600AB^2 - 3300A^2B \]
\[ z = 1550A^3 + 850B^3 - 4650AB^2 - 2550A^2B \]
\[ w = 250A^3 + 1750B^3 - 750AB^2 - 5250A^2B \]
\[ p = 25A^2 + 25B^2 \]

OBSERVATIONS:

1. \( -\frac{x(A,A)}{100A^3} + 3 \) = Kynea number

2. \( -\frac{z(A,A)}{100A^3} + 1 \) = Carol no.

3. \( 5.4(2P_a^3 - 43T_{10,a} - 66\text{Gno}_a) - [y(A,L) + w(A,L)] - 67 = W_a \)

4. \( 4.6P_a^{13} - 2T_{1L,a} - 16\text{Gno}_a - \frac{x(A,L)}{100} + 1 = \text{Mersenne no.} \)

5. \( 5.6P_a^{13} - 2T_{1L,a} - 16\text{Gno}_a - \frac{x(A,L)}{100} - 3 = \text{Thabit - ibn - karrabah no.} \)

PATTERN 4:

Write \( 1 = \frac{(11 + 60i)(11 - 60i)}{61^2} \) (9)

Using (4), (5) and (9) in (6) and proceeding as in pattern 3 we get

\[ u = \frac{1}{61} [-54a^3 + 382b^3 + 162ab^2 - 1146a^2b] \]
\[ v = \frac{1}{61} [382a^3 + 54b^3 - 1146ab^2 - 162a^2b] \]

Since our interest is to find integer solutions, taking \( a = 61A; b = 61B \), in the above equations and proceeding as in earlier patterns we get the distinct non-zero integer solutions of (1) to be

\[ x = 1220488A^3 + 1622356B^3 - 4867068A^2B - 3661464AB^2 \]
\[ y = -1622356A^3 + 1220488B^3 - 3661464A^2B + 4867068AB^2 \]
\[ z = 1019554A^3 + 3043778B^3 - 9131334A^2B - 3058662AB^2 \]
\[ w = -1823290A^3 + 2641910B^3 - 7925730A^2B + 5469870AB^2 \]
\[ p = 3721A^2 + 3721B^2 \]

OBSERVATIONS:

1. \( -\frac{x(A,A)}{71071A^3} - 1 = \text{Mersenne no.} \)

2. \( \frac{y(A,A)}{33489A^3} - 1 = \text{Thabit - ibn - karrabah no.} \)

3. The following expressions represent the Nasty number
   i) \( y(LI) \)
   ii) \( -[z(LI)] + 11250 \)
   iii) \( -[x(LI)] + w(LI) + 10032 \)

4. \( -\frac{z(LI)}{61^2} + 1 \) = Sierpinski no.
IV. CONCLUSION

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the non-homogeneous quintic diophantine equation having five unknowns given by

\[ x^4 - y^4 = 40(z^2 - w^2)p^3 \]

One can also search for other patterns of solutions for the above equation.

V. REFERENCES


