Local Continuity versus Global Uniform Continuity

S. Sumathira
Assistant Professor
Department of Mathematics (SF)
Nehru Memorial College, Puthanampatti, Trichy, India

Abstract:
A function \( f \) is uniformly continuous if, roughly speaking, it is possible to guarantee that \( f(x) \) and \( f(y) \) be as close to each other as we please by requiring only that \( x \) and \( y \) are sufficiently close to each other; unlike ordinary continuity, the maximum distance between \( f(x) \) and \( f(y) \) cannot depend on \( x \) and \( y \) themselves. For instance, any isometry (distance-preserving map) between metric spaces is uniformly continuous. We have the following chain of inclusions for functions over a compact subset of the real line.

Continuously differentiable \( \leq \) Lipschitz continuous \( \leq \) \( \alpha \)-Hölder continuous \( \leq \) uniformly continuous \( = \) continuous

Keywords: Continuously differentiable, Lipschitz continuous, \( \alpha \)-Hölder continuous, uniformly continuous and continuous

1. INTRODUCTION

Every uniformly continuous function between metric spaces is continuous. Uniform continuity, unlike continuity, relies on the ability to compare the sizes of neighbourhoods of distinct points of a given space. Instead, uniform continuity can be defined on a metric space.

2. MAIN RESULT

2.1 Definition: continuously differentiable
A function \( f \) is said to be continuously differentiable if the derivative \( f'(x) \) exists and is itself a continuous function.

For example
\[
 f(x) = \begin{cases} \sin \left( \frac{x}{2} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
\]

2.2 Definition: Uniformly continuous
Given metric spaces \( (X, d_X) \) and \( (Y, d_Y) \), a function \( f: X \to Y \) is called uniformly continuous if for every real number \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( x, y \in X \) with \( d_X(x, y) < \delta \), we have that \( d_Y(f(x), f(y)) < \varepsilon \).

2.2.2 Example
Define \( f(x) = \frac{1}{x} \) and \( g(x) = \sqrt{x} \).

Uniform continuity

2.3 Definition: Lipschitz continuous
Let \( \mathcal{B} \subseteq \mathbb{R} \), \( g: \mathcal{B} \to \mathbb{R} \) is called Lipschitz continuous if there exists a positive real constant \( K \) such that, for all real \( x_1 \) and \( x_2 \),
\[
|g(x_1) - g(x_2)| \leq K|x_1 - x_2|.
\]

2.4 Definition: \( \alpha \)-Hölder continuous
A real or complex-valued function \( g \) on \( d \)-dimensional Euclidean space satisfies a Hölder condition, or is Hölder continuous, when there are no negative real constants \( C, \alpha \), such that
\[
|g(x_1) - g(x_2)| \leq C|x_1 - x_2| ^{\alpha}
\]
The number \( \alpha \) is called the exponent of the Hölder condition.
If \( \alpha = 1 \), then the function satisfies a Lipschitz condition.
If \( \alpha > 0 \), the condition implies the function is continuous.
If \( \alpha = 0 \), the function need not be continuous, but it is bounded.

2.5 Definition: continuous
Let \( A \subseteq \mathbb{R} \), \( f: A \to \mathbb{R} \) is continuous if \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that
\[
|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.
\]

Theorem 2.6
Every continuous function on a open set is uniform continuous

Proof
Let \( M \) and \( N \) are two metric spaces with \( d_M \) and \( d_N \), respectively.
Let \( f: M \to N \) is continuous function
To prove that: \( f \) is uniform continuous
Let \( f \) is function, ie, \( f \) is open.
By the definition of continuous,
Let \( d_M \subseteq d_N \), \( f: d_M \to d_N \) is continuous if \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that
\[
d_M(x, y) < \frac{\delta}{2} \implies d_N(f(x), f(y)) < \varepsilon.
\]
Which implies
\[
d_M(x, y) \leq d_M(x_i, y) + d_M(x, y) < \frac{1}{2}\delta_{x_i} + \delta < \delta_{x_i}
\]
Which implies
\[
d_N(f(x_i), f(y)) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.
\]
Clearly which is uniform continuous.
Hence “every continuous function on a open set is uniformly continuous”

3. CONCLUSION

Motivated by the definition of local continuity versus, we have discussed above uniform continuous investigation.

4. REFERENCES

