Some Fixed Point Theorems using Property E.A. in Fuzzy Metric Spaces

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Abstract:
We establish some common fixed point theorems for weakly compatible maps in fuzzy metric spaces by using E.A. property.

I. INTRODUCTION AND PRELIMINARIES

Zadeh [3] introduced the notion of fuzzy set. Out of many viewpoints of the notion of metric space, here we recall some terminologies, definitions and lemmas from the theory of fuzzy metric spaces. Kramosil and Michalek [5] introduced a fuzzy metric space with some axioms, we call this space as KM fuzzy metric space. Another fuzzy metric space introduced by George and Veeramani [6], we refer this space as GV fuzzy metric space.

Definition 1.1 A continuous t-norm (in the sense of Schweizer and Sklar [4]) is a binary operation * on [0,1] satisfying the following conditions:

(i) * is commutative and associative;
(ii) a * 1 = a for all a ∈ [0,1];
(iii) a * b ≤ c * d whenever a ≤ c and b ≤ d (a, b, c, d ∈ [0,1]);
(iv) the mapping * : [0,1] × [0,1] → [0,1] is continuous.

Definition 1.2 A fuzzy metric space in the sense of Kramosil and Michalek [5] is a triplet (X, M, s) where X is a nonempty set, * is a continuous t-norm and M is a fuzzy set on X² × [0,∞) such that the following axioms hold:

(FM-1) M(x,y,0) = 0 (x,y ∈ X).
(FM-2) M(x,y,t) = 1 ∀ t > 0 iff x = y.
(FM-3) M(x,y,t) = M(y,x,t) (x,y ∈ X, t > 0);
(FM-4) M(x,y,⋅) : [0,∞) → [0,1] is left continuous ∀ x,y ∈ X;
(FM-5) M(x,z,t+s) ≥ M(x,y,t) * M(y,z,s) ∀ x,y,z ∈ X ∀ t,s > 0.

We will refer to these as KM-fuzzy metric space.

Lemma 1.3 For every x,y ∈ X, the mapping M(x,y,⋅) is nondecreasing on (0,∞).

Definition 1.4 A fuzzy metric space in the sense of George and Veeramani [6] is a triplet (X, M, s) where X is a nonempty set, * is a continuous t-norm, M is a fuzzy set on X² × (0,∞) and the following conditions are satisfied for all x,y ∈ X and t,s > 0:

(GV-1) M(x,y,t) > 0;
(GV-2) M(x,y,t) = 1 iff x = y;
(GV-3) M(x,y,t) = M(y,x,t);
(GV-4) M(x,y,⋅) : (0,∞) → [0,1] is continuous;
(GV-5) M(x,z,t+s) ≥ M(x,y,t) * M(y,z,s).

It is clear from (GV-2) and (GV-1) that if x ≠ y, then 0 < M(x,y,t) < 1 ∀ t > 0.

Example 1.5 Let (X, d) be a metric space, a * b = T_M(a, b) and for all x, y ∈ X and t > 0,

M(x,y,t) = t

M(x,y,t) = t + d(x,y).

Then (X, M, s) is a GV-fuzzy metric space, called standard fuzzy metric space induced by (X, d).

Definition 1.6 Let (X, M, s) be a (KM or GV) fuzzy metric space. A sequence (x_n), n ∈ N in X is said to be convergent to x ∈ X if lim_{n→∞} M(x_n,x,t) = 1 for all t > 0. A sequence (x_n) in X is said to be a Cauchy sequence [6,8] if lim_{n→∞} M(x_n,x_n+p,t) = 1 for all t > 0 and p ∈ N.

A fuzzy metric space is called G-complete if every G-Cauchy sequence converges in X.

Lemma 1.7 If (X, M, s) is a KM-fuzzy metric space and (x_n), (y_n) are sequences in X such that x_n → x, y_n → y, then M(x_n,y_n,t) → M(x,y,t) for every continuity point t of M(x,y,⋅).

Fixed point theory in fuzzy metric spaces developed in the paper of Grabiec and Subrahmanyam [11] gave a generalization of Jungack [12] common fixed point theorem for commuting mappings in the fuzzy metric spaces. The existence of a common fixed point requires few conditions on continuity of the maps, G-completeness of the space. Aamri and Moutawakil recently introduced the concept of E.A. property in a metric space. The property is as follows.

Definition 1.8 Two self-maps A and S of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence (x_n) in X such that lim_{n→∞} A x_n = lim_{n→∞} S x_n = t for some t ∈ X.

Similarly, it is said that two self-maps A and S of a fuzzy metric space (X, M, s) satisfy E.A. property, if there exist a sequence (x_n) in X and z ∈ X such that {A x_n} and {S x_n} converge to z in the sense of definition 1.6.

Definition 1.9 Two maps f and g are said to be weakly compatible if they commute at their coincidence points [12]. Some common fixed point theorems in fuzzy metric spaces by E.A. property under weak compatibility recently obtained in [16].

The completeness of metric space can be replaced by more natural condition of closeness of the range with the help of E.A. property.
We have worked in fuzzy metric spaces in the sense of Kramosil and Michalek as well as in the sense of George and Veeramani.

We need to involve the class $\Phi$ of all mappings $\phi: [0,1] \rightarrow [0,1]$ satisfying the following properties:

(i) $\phi$ is continuous and non-decreasing on $[0,1]$;

(ii) $\phi(x) > x$ for all $x \in (0,1)$.

We note that if $\phi \in \Phi$, then $\phi(1) = 0$ and that $\phi(x) \leq x$, $\forall x \in [0,1]$.

II. Main Results

**Theorem 2.1** Let $(X, M, *)$ be a $K_M$-fuzzy metric space satisfying the following property:

For all $x, y, z \in X, x \neq y, \exists t > 0 \colon M(x, y, t) < 1$ and $f, g$ be weakly compatible self-maps of $X$ such that, for some $\phi \in \Phi$,

$$M(fx, fy, t) \geq \phi(M(x, y, t))$$

for all $t > 0$. Hence, it must be a tric space. The above inequality.

Then, by taking the limit as $n$ tends to infinity in

$$M(fx_n, fu, t) \geq \phi(M(x_n, u, t))$$

we obtain the inequality

$$M(fu, gu, s) \geq \phi(M(fu, gu, s)).$$

Now, if $fu \neq gu$, then $0 < M(fu, gu, s) < 1$, that is, $\phi(M(fu, gu, s)) > M(fu, gu, s)$, which contradicts the above inequality. Hence proves that $M(fu, gu, s) = 1$, due to (GV-2), $fu = gu$. By denoting $fu(= gu)$ by $z$, we can say that $fz = gz$. Now we prove that $f(z) = z$. We know that $M(fz, z, s) = M(fz, fu, s) = \phi(M(fu, gu, s))$.

**Proof:** We can find a sequence $(x_n)$ in $X$ and a point $u \in X$ such that $lim_{n \rightarrow \infty} x_n = lim_{n \rightarrow \infty} gu_n = gu$.

It is clear that $\phi (M(fu, gu, s)) > M(fu, gu, s)$. This contradicts the above inequality. Hence proves that $M(fu, gu, s) = 1$, due to (GV-2).

To prove uniqueness of the common fixed point, let us suppose that $w$ is a common fixed point of $f$ and $g$ and $w \neq z$. Then $0 < M(w, z, t) < 1$ for all $t > 0$ and thus $\phi(M(w, z, t)) > M(w, z, t), \forall t > 0$.

On the other hand, $M(z, w, s) = M(fz, gw, s) \geq \phi(M(gz, gw, t)) = \phi(M(z, w, t))$.

For all $t > 0$, which is a contradiction. Therefore, the equality $fu = gu$ holds.

Denote $fu (= gu)$ by $z$. From weak compatibility, it follows $fz = gz$, hence

$$M(fz, z, t) = M(fu, gu, t) = \phi(M(z, w, t)) \quad (t > 0).$$

As above we obtain $fz = z$, that is, $z$ is a common fixed point for $f$ and $g$. To conclude the proof, let us suppose that $w$ is also a common fixed point. If $w \neq z$, then $0 < M(z, w, t) < 1$ for some $t > 0$, hence $M(z, w, t) < \phi(M(z, w, t))$.

On the other hand, $M(z, w, t) = M(fz, gw, t) \geq \phi(M(gz, gw, t)) = \phi(M(z, w, t))$,

for all $t > 0$, which is a contradiction. Therefore, it must be the case that $z = w$.

**Example 2.2** Let $X = (0, \infty)$ and, for each $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{\min(x, y)}{\max(x, y)}.$$

It is clear that $(X, M, T_M)$ is a $G_M$-fuzzy metric space. The mappings $f, g$ defined on $X$ through $fx = x$ and $gx = x$ satisfy all the conditions of the above corollary, with $\phi(t) = t (t \in [0,1])$. Their common fixed point is $x = 1$.

**Theorem 2.3** Let $(X, M, *)$ be a $G_M$-fuzzy metric space and $f, g$ be weakly compatible self-maps of $X$ such that, for some $\phi \in \Phi$ and some $s > 0$,

$$M(fx, fy, s) \geq \phi(M(fx, fy, s)) \geq M(fx, fy, s)$$

for all $x, y \in X$. If $f$ and $g$ satisfy E.A. property and the range of $g$ is a closed subspace of $X$, then $f$ and $g$ have a unique common fixed point.

**Proof:** We can find a sequence $(x_n)$ in $X$ and a point $u \in X$ such that $lim_{n \rightarrow \infty} x_n = lim_{n \rightarrow \infty} gu_n = gu$.

We need to involve the class $\Phi$ of all mappings $\phi: [0,1] \rightarrow [0,1]$ satisfying the following property:

(i) $\phi$ is continuous and non-decreasing on $[0,1]$;

(ii) $\phi(x) > x$ for all $x \in (0,1)$.

We note that if $\phi \in \Phi$, then $\phi(1) = 0$ and that $\phi(x) \leq x$, $\forall x \in [0,1]$.

Now, if $fu \neq gu$, then $0 < M(fu, gu, s) < 1$, that is, $\phi(M(fu, gu, s)) > M(fu, gu, s)$. This contradicts the above inequality. Hence proves that $M(fu, gu, s) = 1$, due to (GV-2), $fu = gu$. By denoting $fu(= gu)$ by $z$, we can say that $fz = gz$. Now we prove that $f(z) = z$. We know that $M(fz, z, s) = M(fz, fu, s) = \phi(M(fu, gu, s))$.

If $f(z) \neq z$ then, from (GV-2), $0 < M(fz, z, t) < 1$ for all $t > 0$ and therefore $\phi(M(fz, z, t)) > M(fz, z, t) \quad (t > 0)$.

In particular, $\phi(M(fz, z, s)) > M(fz, z, s)$, contradicting the above inequality. Thus, we obtain that $fz = z$, hence $z$ is a common fixed point for $f$ and $g$.

To prove uniqueness of the common fixed point, let us suppose that $w$ is a common fixed point of $f$ and $g$ and $w \neq z$. Then $0 < M(w, z, t) < 1$ for all $t > 0$ and thus $\phi(M(w, z, t)) > M(w, z, t), \forall t > 0$.

On the other hand, $M(z, w, s) = M(fz, gw, s) \geq \phi(M(gz, gw, t)) = \phi(M(z, w, t))$.

For all $t > 0$, which is a contradiction. Hence proves the theorem.

**Example 2.4** Consider the space $(X, M, T_M)$, where $X = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$,

$$M(x, y, t) = \frac{t}{t + |x - y|} \quad (t > 0)$$

and the mappings $f, g : X \rightarrow X, fx = 1, \forall x \in X, gx = 1$ if $x = 1$.

if $x \neq 1$. If $\phi: [0,1] \rightarrow [0,1], \phi(t) = \sqrt{t}$, then all the conditions of the preceding theorem are satisfied. The common fixed point of $f$ and $g$ is $x = 1$. 

Theorem 2.5 Let \((X,M,\ast)\) be a KM-fuzzy metric space satisfying the following property: for all \(x,y \in X, x \neq y, \exists t > 0 : 0 < M(x,y,t) < 1\) and \(f, g\) and \(h\) be weakly compatible self-maps of \(X\) such that, for some \(\phi \in \Phi, M(fx,gy, t)\geq \phi(M(hx,hy, t),M(fx, hx, t),M(gy, hy, t))\)

\(\forall x \in X\) and \(t \in\). Therefore the equality

\[ M(fx_n,gu,t),M(gx_n,hx_n,t)\geq \phi(M(fx_n,gu,t),M(gx_n,hx_n,t)) \]

Thus, \(x_n\) is a common fixed point of \(f, g, h\). Weak compatibility of mappings gives \(hx_n = f(x_n) = g(x_n)\), for all \(n \in \mathbb{N}\).

Proof: Since \(f, g\) and \(h\) satisfy the E.A. property, there exists a sequence \((x_n)\) in \(X\) such that \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} h(x_n) = p \in X\). Then, by the taking the limit as \(n \to \infty\) we obtain the inequality

\[ M(hu,gu,t) \geq \phi(M(\mu, h\mu, t), M(gu, h\mu, t)) \]

Taking limit as \(n \to \infty\) we get

\[ M(hu,gu, t) \geq \phi(M(\mu, h\mu, t)) \]

Now, if \(hu \neq gu\), then \(0 < M(hu,gu, t) < 1\), for some \(s > 0\). As \(M(\mu, h\mu, t)\) is nondecreasing, that is it has at most countable points of discontinuity. Suppose \(s\) is a continuity point of \(M(\mu, h\mu, t)\).

Then by condition \(\phi_2\) we have

\[ \phi(M(\mu, s), M(fu, s), M(gu, s)) \]

which is contradiction. Therefore, the equality \(hu = gu\) holds. Weak compatibility of functions gives \(hz = gz\). Now consider

\[ M(gx_n,hx_n,t),M(fx_n,hx_n,t),M(gu,hu,t),M(fu,hu,t),M(gu,hx_n,t),M(fu,hx_n,t)) \]

Taking limit as \(n \to \infty\) we get

\[ M(hu,gu,t) \geq \phi(M(\mu, h\mu, t)) \]

Now, if \(hu \neq fu\), then \(0 < M(hu,fu, t) < 1\), for some \(t > 0\). As \(M(\mu, h\mu, t)\) is nondecreasing, that is it has at most countable points of discontinuity. Suppose \(s\) is a continuity point of \(M(\mu, h\mu, t)\).

Then by condition \(\phi_2\) we have

\[ \phi(M(hu, fu, t), M(fu, t), M(gu, t)) \]

which is contradiction. Hence \(hu = fu\). Weak compatibility of functions gives \(hz = fz\). Thus \(fz = gz = hz = z\). Uniqueness of a common fixed point is followed by previous theorem.

Theorem 2.6 Let \((X,M,\ast)\) be a KM-fuzzy metric space satisfying the following property: for all \(x,y \in X, x \neq y, \exists t > 0 : 0 < M(x,y,t) < 1\) and \(f, g\) and \(h\) be weakly compatible self-maps of \(X\) such that, for some \(\phi \in \Phi, M^2(fx,gy, t)\geq \phi(M(hx,hy, t),M(fx, hx, t),M(gy, hy, t))).

\(\forall x \in X\) and \(t \in\). Therefore the equality

\[ M^2(hu,gu,t) \geq \phi(M(gu, hu, t), M(gu, hu, t)) \]

Now, if \(hu \neq gu\), then \(0 < M(hu,gu, s) < 1\), for some \(s > 0\).

Clearly

\[ 0 < M(hu,gu, t) \geq M^2(2^*(gu, hu, t)) \]

As \(M(gu,hu,t)\) is left continuous and \(M(gu,hu,t)\) is nondecreasing, that is it has at most countable points of discontinuity. Suppose \(s\) is a continuity point of \(M(gu,hu,t)\).

Then by condition \(\phi_2\), we have

\[ \phi(M(gu, s), M(fu, s), M(gu, fu, s)) \]

which is contradiction. Therefore the equality \(hu = gu\) holds. Weak compatibility of functions gives \(hz = gz\). Similarly, we can show that \(hz = fz\). To prove that the common fixed point is unique, let us suppose that \(w\) is a common fixed point of \(f, g, h\) and \(w \neq z\). Then \(0 < M(w, z, t) < 1\) for all \(t > 0\) and thus

\[ \phi(M(w, z, t)) \]

On the other hand, we know that

\[ M^2(z, w, s) = M^2(fz, w, s) \]

Using \(\phi\) we get

\[ \phi(M(hz, hw, s), M(fz, hw, s), M(gw, hw, s), M(fw, hw, s), M(gw, hw, s), M(fw, hw, s)) \]

Then by taking limit as \(n \to \infty\) we get

\[ \phi(M(z, w, s), M(z, w, s), M(z, w, s), M(z, w, s)) \]

\[ \phi(M(z, w, s), M(z, w, s), M(z, w, s), M(z, w, s)) \]

which is a contradiction. This concludes the proof.

Theorem 2.7 Let \((X,M,\ast)\) be a GV-fuzzy metric space and \(f, g\) and \(h\) be weakly compatible self-maps of \(X\) such that, for some \(\phi \in \Phi\) and some \(t > 0\).

\[ M(fx,gy, t)\geq \phi(M(hx,hy, t),M(fx, hx, t),M(gy, hy, t),M(fy, hy, t),M(gy, hy, t))) \]

If \(f, g\) and \(h\) satisfy E.A. property and the range of \(f, g, h\) is a closed subspace of \(X\), \(f \cup g \subseteq h(X)\) and \(h(X)\) is a closed subspace of \(X\), then \(f, g, h\) and \(h\) have a unique common fixed point.

Proof: We can have a sequence \((x_n)\) in \(X\) and \(t \in\) such that

\[ \lim_{n \to \infty} x_n = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} h(x_n) = p \in X\].

Then by taking limit as \(n \to \infty\) we obtain the inequality

\[ M(hp, gp, t) \geq \phi(M(gp, gp, t)) \]

Now, if \(gp \neq hp\), then \(0 < M(hp, gp, t) < 1\), therefore

\[ \phi(M(gp, gp, t)) \geq M(hp, gp, t) \]

which contradicts above inequality, proving the fact that \(M(hp, gp, t) = 1\).

Using (GV-2) we get \(hp = gp\). As mappings \(g\) and \(h\) are weakly compatible, we have \(hp = gp\).

Now, let us prove \(fp = hp\). Consider

\[ M(gx_n, fp, t)\geq \phi(M(gx_n, hp, t),M(gp, hp, t)) \]

which is left continuous and \(M(gx_n, hp, t)\) is nondecreasing, that is it has at most countable points of discontinuity. Suppose \(s\) is a continuity point of \(M(gx_n, hp, t),M(gp, hp, t))\).

Then by taking limit as \(n \to \infty\) we get

\[ M(hp, fp, t) \geq \phi(M(fp, hp, t)) \]

Now, if \(fp \neq hp\), then \(0 < M(fp, hp, t) < 1\), therefore

\[ \phi(M(hp, hp, t)) \geq M(hp, fp, t), \]

which contradicts above inequality, proving the fact that \(M(hp, hp, t) = 1\).

Using (GV-2) we get \(hp = fp\). Hence \(fp = gp = hp\). Weak compatibility of mappings gives \(fz = gz = hz = z\). To prove uniqueness, let us assume \(w \in X\) is such that \(fw = gw = hu = w, w \neq z\).
Then $0 < M(w, z, t) < 1$ for all $t > 0$. 
\[ \phi(M(z, w, t)) > M(z, w, t) \quad \forall t > 0. \]
On the contrary,
\[ M(z, w, s) = M(fz, fw, s) \]
\[ \geq \phi(\min\{ M(gz, gw, s), M(fw, gw, s), M(fz, gw, s) \}) \]
\[ = \phi(M(z, w, s)). \]
Hence the uniqueness followed.

**Theorem 2.8** Let $(X, M, \ast)$ be a GV-fuzzy metric space satisfying the following property:
for all $x, y \in X, x \neq y, \exists t > 0: 0 < M(x, y, t) < 1$ and $f, g$ and $h$ be weakly compatible self-maps of $X$ such that, for some $\phi \in \Phi,$
\[ M^2([fx, gy, t] \cap \{ \min \{ M(hx, hy, t) - M(fx, hy, t), M(gy, hy, t) \} \}) \]
\[ \geq \phi(M([fx, gy, t] \cap \{ \min \{ M(hx, hy, t) - M(fx, hy, t), M(gy, hy, t) \} \})). \]
Now, if $h \neq f, g$, then $0 < M(hu, gu, s) < 1$, for some $s > 0$. As $M(gu, hu, \cdot)$ is left continuous and $M(gu, hu, \cdot)$ is nondecreasing, that is it has at most countable points of discontinuity, Suppose $s$ is a continuity point of $M(gu, hu, \cdot).$ Then by condition (GV-2) we have
\[ \phi(M(gu, hu, s)) > M(gu, hu, s), \]
which is contradiction. Therefore, the equality $hu = gu$ holds. Weak compatibility of functions gives $hz = gz$. Now consider,
\[ M(gx, fu, t) \geq \phi(\min \{ M(hx, hu, t), M(gx, hu, t), M(fu, hu, t), M(gx, hu, t), M(fu, hu, t) \}). \]
Taking limit as $t$ tends to infinity, we get
\[ M(hu, fu, t) \geq \phi(M(fu, hu, t)). \]
Now, if $h \neq f, g$, then $0 < M(hu, fu, t_0) < 1$, for some $t_0 > 0$. As $M(hu, fu, \cdot)$ is left continuous and $M(hu, fu, \cdot)$ is nondecreasing, that is it has at most countable points of discontinuity, Suppose $s$ is a continuity point of $M(hu, fu, \cdot).$ Then by condition (GV-2), we have
\[ \phi(M(hu, fu, t_0)) > M(hu, fu, t_0). \]
Which is contradiction. Hence $hu = fu.$ Weak compatibility of functions gives $hz = Fz$. Thus $Fz = gz = hz = z$. Uniqueness of a common fixed point is followed as in previous theorem.

### III. REFERENCES
